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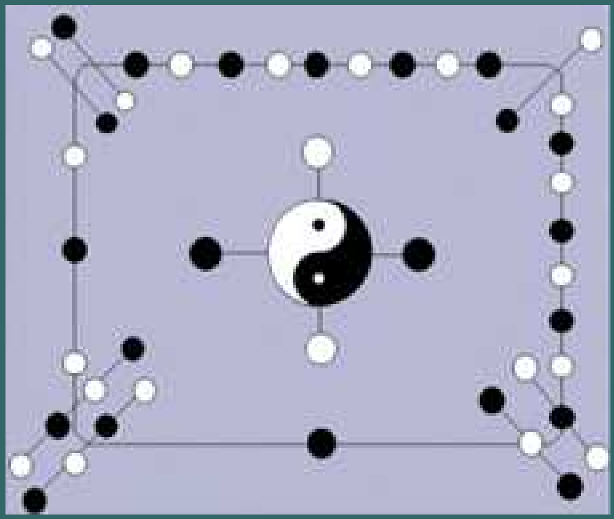

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(International Book Series)

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Duality Theorems of Multiobjective Generalized Disjunctive Fuzzy Nonlinear Fractional Programming

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Abstract: This paper is concerned with the study of duality conditions to convex-concave generalized multiobjective fuzzy nonlinear fractional disjunctive programming problems for which the decision set is the union of a family of convex sets. The Lagrangian function for such problems is defined and the Kuhn-Tucker Saddle and Stationary points are characterized. In addition, some important theorems related to the Kuhn-Tucker problem for saddle and stationary points are established. Moreover, a general dual problem is formulated together with weak; strong and converse duality theorems are proved.

Key Words: Generalized multiobjective fractional programming; Disjunctive programming; Convexity; Concavity; fuzzy parameters Duality.

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§1. Introduction

Fractional programming models have been became a subject of wide interest since they provide a universal apparatus for a wide class of models in corporate planning, agricultural planning, public policy decision making, and financial analysis of a firm, marine transportation, health care, educational planning, and bank balance sheet management. However, as is obvious, just considering one criterion at a time usually does not represent real life problems well because almost always two or more objectives are associated with a problem. Generally, objectives conflict with each other; therefore, one cannot optimize all objectives simultaneously. Non-differentiable fractional programming problems play a very important role in formulating the set of most preferred solutions and a decision maker can select the optimal solution.

Chang in [8] gave an approximate approach for solving fractional programming with absolute-value functions. Chen in [10] introduced higher-order symmetric duality in non-differentiable multiobjective programming problems. Benson in [6] studied two global optimization problems, each of which involves maximizing a ratio of two convex functions, where at least one of the two convex functions is quadratic form. Frenk in [12] gives some general results of the above Benson problem. The Karush-Kuhn-Tucker conditions in an optimization problem with interval-valued objective function are derived by Wu in [29].

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Balas introduced Disjunctive programs in [3, 4,]. The convex hull of the feasible points has been characterized for these programs with a class of problems that subsumes pure mixed integer programs and for many other non-convex programming problems in [5]. Helbig presented in [17, 18] optimality criteria for disjunctive optimization problems with some of their applications. Gugat studied in [15, 16] an optimization a problem having convex objective functions, whose solution set is the union of a family of convex sets. Grossmann proposed in [14] a convex nonlinear relaxation of the nonlinear convex disjunctive programming problem. Some topics of optimizing disjunctive constraint functions were introduced in [28] by Sherali. In [7], Ceria studied the problem of finding the minimum of a convex function on the closure of the convex hull of the union of a finite number of closed convex sets. The dual of the disjunctive linear fractional programming problem was studied by Patkar in [25]. Eremin introduced in [11] disjunctive Lagrangian function and gave sufficient conditions for optimality in terms of their saddle points. A duality theory for disjunctive linear programming problems of a special type was suggested by Gon?alves in [13].

Liang In [21] gave sufficient optimality conditions for the generalized convex fractional programming. Yang introduced in [30] two dual models for a generalized fractional programming problem. Optimality conditions and duality were considered in [23] for nondifferentiable, multiobjective programming problems and in [20, 22] for nondifferentiable, nonlinear fractional programming problems. Jain et al in [19] studied the solution of a generalized fractional programming problem. Optimality conditions in generalized fractional programming involving nonsmooth Lipschitz functions are established by Liu in [23]. Roubi [26] proposed an algorithm to solve generalized fractional programming problem. Xu [31] presented two duality models for a generalized fractional programming and established its duality theorems. The necessary and sufficient optimality conditions to nonlinear fractional disjunctive programming problems for which the decision set is the union of a family of convex sets were introduced in [1]. Optimality conditions and duality for nonlinear fractional disjunctive minimax programming problems were considered in [2]. In this paper we define the Langrangian function for the nonlinear generalized disjunctive multiobjective fractional programming problem and investigate optimality conditions. For this class of problems, the Mond-Weir and Schaible type of duality are proposed. Weak, strong and converse duality theorems are established for each dual problem.

§2. Problem Statement

Assume that $N = \{1, 2, \cdots, p\}$ and $\mathcal{K} = \{1, 2, \cdots, q\}$ are arbitrary nonempty index sets. For $i \in N$, let $g^i_j : \mathbf{R}^n \to \mathbf{R}$ be a vector map whose components are convex functions, $g^i_j(x) \le 0$, $1 \le j \le m$. Suppose that $f^{ik}_r,\ h^{i+m+k}_r :\ \mathbf{R}^{n+q} \to \mathbf{r}$ are convex and concave functions for $i \in N,\ k \in \mathcal{K},\ r = 1, \cdots, s$ respectively, and $h^{ik}_r(x, \widetilde{b}_r) > 0$. Here, these $\widetilde{a}_r, \widetilde{b}_r,\ r = 1, 2, \cdots, m$ represent the vectors of fuzzy parameters in the objectives functions. These fuzzy parameters are assumed to be characterized as fuzzy numbers [4].

We consider the generalized disjunctive multiobjective convex-concave fractional program

problem as in the following form:

$$\text{GDFFVOP}(i) \qquad \inf_{x\in Z_i}\max_{k\in K}\left\{\frac{f_r^{ik}(x,\widetilde{a}_r)}{h_r^{ik}(x,\widetilde{b}_r)},\quad r=1,2,\cdots,s\right\}, \tag{1}$$

$$\text{Subject to} \qquad \mathrm{x}\in \mathrm{Z_i},\ \mathrm{i}\in \mathrm{N}, \tag{2}$$

where $Z_i = \{x \in \mathbf{R}^n :\ g_j^i(x) \le 0,\ j = 1, 2, \cdots, m\}$. Assume that $Z_i \neq \emptyset$ for $i \in N$.

Definition 1([1]) *The α-level set of the fuzzy numbers $\widetilde{a}$ and $\widetilde{b}$ are defined as the ordinary set $S_\alpha(\widetilde{a},\widetilde{b})$ for which the degree of their membership functions exceeds level α:*

$$S_\alpha(\widetilde{a},\widetilde{b}) = \{(a,b)\in \mathbf{R}^{2m} | \mu_{ar}(a_r)\ge\alpha,\ r=1,2,\cdots,m\}.$$

For a certain degree of α, the GDFVOP(i) problem can be written in the ordinary following form [11].

Lemma 1([7]) *Let $\alpha^k, \beta^k,\ k \in \mathcal{K}$ be real numbers and $\alpha^k > 0$ for each $k \in K$. Then*

$$\max_{k\in K}\frac{\beta^k}{\alpha^k} \ge \frac{\sum\limits_{k\in K}\beta^k}{\sum\limits_{k\in K}\alpha^k}. \tag{3}$$

By using Lemma 1 and from [9] The generalized multiobjective fuzzy fractional problem GDFFVOP(i) may be reformulated [3] as in the following two forms:

GDFFNLP(i, t, α):

$$\inf_{i\in N}\inf_{x\in Z_i(S)}\left\{\frac{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K} t^k f_r^{ik}(x,a_r)}{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K} t^k h_r^{ik}(x,b_r)},\ (a_r,b_r)\in S_\alpha(\widetilde{a},\widetilde{b}), r=1,2,\cdots,m\right\}, \tag{4}$$

where $t^k \in \mathbf{R}_+^q$. Denote by

$$M_i = \inf_{x\in Z_i}\frac{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K} t^k f_r^{ik}(x,a_r)}{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K} t^k h_r^{ik}(x,b_r)},\ (a_r,b_r)\in S_\alpha(\widetilde{a},\widetilde{b}), r=1,2,\cdots,m$$

the minimal value of GDFFNLP(i, t, α), and let

$$P_i = \left\{x\in Z_i:\ \frac{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K} t^k f_r^{ik}(x,a_r)}{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K} t^k h_r^{ik}(x,b_r)} = M_i,\ i\in N\right\}$$

be the set of solutions of GDFFNLP(i, t, α). The generalized multiobjective disjunctive fuzzy fractional programming problem is formulated as:

$$\text{GDFFNLP}(t,\alpha): \qquad \inf_{i\in N}\inf_{x\in Z}\left\{\frac{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K} t^k f_r^{ik}(x,a_r)}{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K} t^k h_r^{ik}(x,b_r)}\right\}, \tag{5}$$

where $t^k \in \mathbf{R}^q_+, k \in \mathcal{K}$ and $Z = \bigcup_{i \in N} Z_i$ is the feasible solution set of problem GDFFNLP(t,α). For problem GDFFNLP(t,α), we assume the following sets:

(I) $M = \inf_{i \in N} M_i$ is the minimal value of GDFFNLP(t,α).

(II) $Z^* = \left\{ x \in Z : \exists i \in I(X),\ \inf_i \dfrac{\sum_{r=1}^{s}\sum_{k=1}^{K} t^k f_r^{ik}(x,a_r)}{\sum_{r=1}^{s}\sum_{k=1}^{K} t^k h_r^{ik}(x,b_r)} = M \right\}$ is set of these of solutions on the problem GDFFNLP(t,α), where $I = \{i \in I' : x \in Z\}$, $I' = \{i \in N : \ Z^* \neq \emptyset\}$ and $I' = \{1, 2, \cdots, a\} \subset N$. Problem GDFFVOP$(t,\alpha)$ may be reformulated in the following form:

GDFFNLP(t,α,d):

$$\inf_{i \in I} \inf_{x \in Z} \left\{ F^i(x,t,d^i,a,b) = \sum_{r=1}^{s}\sum_{k=1}^{K} t^k f_r^{ik}(x,a_r) - d^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k h_r^{ik}(x,b_r) \right\}, \tag{6}$$

where

$$d^i = \frac{\sum_{r=1}^{s}\sum_{k=1}^{K} t^k f_r^{ik}(x,a_r)}{\sum_{r=1}^{s}\sum_{k=1}^{K} t^k h_r^{ik}(x,b_r)} > 0, \quad i \in I.$$

We define the Lagrangian functions of problems GDFFNLP(t,α,d) and GDFFNLP(t,α) [21, 24, and 25] in the following forms:

$$GL^i(x,\lambda^i,a,b) = F^i(x,t,d^i,a,b) + \lambda \sum_{j=1}^{m} \lambda_j^i g_j^i(x) \tag{7}$$

and

$$L^i(x,u,\lambda^i,a,b) = \frac{u^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k f_r^{ik}(x,a_r) + \sum_{j=1}^{m} \lambda_j^i g_j^i(x)}{u^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k h_r^{ik}(x,b_r)}, \tag{8}$$

where $\lambda_j^i \geq 0$ and $u^i \geq 0$, $i \in I$ are Lagrangian multipliers. Then the Lagrangian functions $GL(x,\lambda,a,b)$ and $L(x,u,\lambda,a,b)$ of GDFFNLP(t,α,d) are defined by:

$$GL(x,\lambda,a,b) = \inf_{i \in I} GL^i(x,\lambda^i,a,b) = \inf_{i \in I} \left\{ F^i(x,t,d^i,a,b) + \sum_{j=1}^{m} \lambda_j^i g_j^i(x) \right\} \tag{9}$$

and

$$L(x,u,\lambda,a,b) = \inf_{i \in I} L^i(x,u,\lambda^i,a,b) = \inf_{i \in I} \left\{ \frac{u^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k f_r^{ik}(x,a_r) + \sum_{j=1}^{m} \lambda_j^i g_j^i(x)}{u^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k h_r^{ik}(x,b_r)} \right\}, \tag{10}$$

where $x \in Z$, $t^k \in \mathbf{R}^q_+$, $u \in \mathbf{R}^q_+$ and $\lambda \in \mathbf{R}^q_+$ are Lagrangian multipliers, respectively.

§3. Optimality Theorems with Differentiability

Definition 3.1 *A point* $(x^0, \lambda^0, a^), b^))$ *in* R^{n+p+2m} *with* $\lambda^) \geq 0$ *is said to be a GL-saddle point of problem GDFFNLP*(t, α, d) *if and only if*

$$GL(x^0, \lambda, a, b) \leq GL(x^0, \lambda^0, a^0, b^0) \leq GL(x, \lambda^0, a^0, b^0) \tag{11}$$

for all with $x \in \mathbf{R}^{n+p}$ *and* $\lambda \in \mathbf{R}^m_+$.

Definition 3.1 *A point* $({}^0, u^0, \lambda^0)$ *in* $\mathbf{R}^{n+p+m}$, *with* $u^0 \geq 0$ *and* $\lambda^0 \geq 0$ *is said to be an L-saddle point of problem GDFFNLP*(t, α) *if and only if*

$$L(x^0, \lambda, a, b) \leq L(x^0, \lambda^0, a^0, b^0) \leq L(x, \lambda^0, a^0, b^0) \tag{12}$$

for all with $x \in \mathbf{R}^{n+p}$, $u \in \mathbf{R}^m_+$ *and* $\lambda \in \mathbf{R}^m_+$.

The proof of the following theorems follows as in [3].

Theorem 3.1(Sufficient Optimality Criteria) *If for* $d^{0i} \geq 0$ *the point* $(x^0, u^0, \lambda^0, a^0, b^0)$ *is a saddle point of* $GL(x, \lambda, a, b)$ *and* $F^i(x, t, d^{0i}, a^0, b^0)$, $g^i_j(x)$ *are bounded and convex functions. Then* x^0 *is a minimal solution for the problem GDFFNLP*(t, d).

Corollary 3.1 *If the point* $(x^0, u^0, \lambda^0, a^0, b^0)$ *is a saddle point of* $L(x, u, \mu)$ *and* $F^i(x, t, d^i, a^0, b^0)$, $g^i_j(x)$ *are bounded and convex functions. Then* x^0 *is a minimal solution for the problem GDFFNLP*(t, α).

The proof is follows similarly as proof of Theorem 3.1.

Assumption 3.1 Let $F^i(x, y, d^i, a, b) = 0$ be a convex function on $Conv\ Z$ ($Z = \bigcup\limits_{i \in I}$). If for all $x \in Conv\ Z$, the functions $F^i(x, t^0, d^{0i}, a^0, b^0) - F^i(x^0, t^0, d^{0i}, a^0, b)$, $x^0 \in Conv\ Z$, $i \in I$, $t^0 \in \mathbf{R}^q_+$ and $(a^0, b^0) \in \mathbf{R}^{2m}$ are bounded, then $\inf\limits_{i \in I} \left\{F^i(x, t^0, d^{0i}, a^0, b) - F^i(x^0, t^0, d^{0i}, a^0, b)\right\}$ is a convex function on $Conv\ Z$.

Proposition 3.1 *Under the Assumption 3.1, and if the system*

$$\left.\begin{array}{l} \inf\limits_{i \in I} F^i(x, t^0, d^{0i}, a^0, b) - F^i(x^0, t^0, d^{0i}, a^0, b^0) < 0, \\ g^i_j(x) \leq 0 \ \textit{for at least one}\ i \in I \end{array}\right\}$$

has no solution on $Conv\ Z$, *then* $\exists \lambda^0 \in \mathbf{R}_+$, $\lambda^{0i} \in \mathbf{R}^m_+$, $(\lambda^0, \lambda^{0i}) \geq 0$ *and* $t^0 \in \mathbf{R}^q_+$ *such that*

$$\mu^0 \inf_{i \in i} F^i(x, t^0, d^{0i}, a^0, b^0) + \inf_{i \in i} \sum_{j=1}^{m} \mu^{0i}_j g^i_j(x) \geq 0$$

for $\forall x \in Conv\ Z$.

Corollary 3.2 *With Assumption 3.1,* $g^i_j(x)$, $i \in I$, $j = 1, 2, \cdots, m$ *satisfy the CQ and* x^0 *is an optimal solution of problem GDFFNLP*(t, α), *then there exists* $u^0 \geq 0$ *and* $\lambda^0 \geq 0$ *such that* $(x^0, t^0, \lambda^0, a^0, b^0)$ *is a saddle point of* $L(x^0, t^0, \lambda^0, a^0, b^0)$.

The proof is follows similarly as proof of Theorem 3.2.

§4. Optimality Theorems without Differentiability

Definition 4.1 *The point* $(x^0, \lambda^0, a^0, b^0)$, $x^0 \in x \in \mathbf{R}^{n+p}$, $\lambda^0, a^0, b^0 \in \mathbf{R}^{3m}$, *if they exist such that*

$$\nabla_x GL(x^0, \lambda^0, a^0, b^0) \geq 0, \qquad x^0 \nabla_x GL(x^0, \lambda^0, a^0, b^0) = 0, \tag{13}$$

$$\nabla_{\lambda x} GL(x^0, \lambda^0, a^0, b^0) \geq 0, \qquad \lambda^0 \nabla_\lambda GL(x^0, \lambda^0, a^0, b^0) = 0, \tag{14}$$

$$\sum_{j=1}^{m} \lambda_j^{0i} g_j^i(x^0) = 0, \quad \lambda_j^i \geq 0, \qquad i \in I, \quad j = 1, 2, \cdots, m. \tag{15}$$

is could Kuhn- Tucker stationary point of problem GDFFNLP(t^0, α^0, d^0). *Or, equivalently,*

$$\nabla_x \inf_{i \in I} \left\{ F^i(x^0, t^0, d^{0i}, a^0, b^0) + \lambda_j^{0i} g_j^i(x^0) \right\} = 0, \quad i \in I, \tag{16}$$

$$g_j^i(x^0) \leq 0, \quad i \in I, \quad j = 1, 2, \cdots, m, \tag{17}$$

$$\sum_{j=1}^{m} \lambda_j^{0i} g_j^i(x^0) = 0, \quad \lambda_j^i \geq 0, \qquad i \in I, \quad j = 1, 2, \cdots, m. \tag{18}$$

Definition 4.2 *The point* $(x^0, u^0, \lambda^0, a^0, b^0)$, $x \in \mathbf{R}^{n+p+2m}$, $u \in \mathbf{R}_+^q$ *and* $\lambda \in \mathbf{R}_+^m$, *if they exist such that*

$$\nabla_x L(x^0, u^0, \lambda^0, a^0, b^0) \geq 0, \qquad x^0 \nabla_x L(x^0, u^0, \lambda^0, a^0, b^0) = 0, \tag{19}$$

$$\nabla_u L(x^0, u^0, \lambda^0, a^0, b^0) \geq 0, \qquad u^0 \nabla_\lambda L(x^0, u^0, \lambda^0, a^0, b^0) = 0, \tag{20}$$

$$\nabla_\mu L(x^0, u^0, \lambda^0, a^0, b^0) \geq 0, \qquad \mu^0 \nabla_\lambda L(x^0, u^0, \lambda^0, a^0, b^0) = 0, \tag{21}$$

$$\sum_{j=1}^{m} \lambda_j^{0i} g_j^i(x^0) = 0, \quad \lambda_j^i \geq 0, \qquad i \in I, \quad j = 1, 2, \cdots, m. \tag{22}$$

is could Kuhn- Tucker stationary point of problem GDFFNLP(t^0, α^0). *Or, equivalently,*

$$\nabla_x \inf_{i \in I} \left\{ \frac{u^{0i} \sum\limits_{r=1}^{s} \sum\limits_{k=1}^{K} t^{0k} f_r^{ik}(x^0, a^0) + \sum\limits_{j=1}^{m} \lambda_j^{0i} g_j^i(x^0)}{u^{0i} \sum\limits_{r=1}^{s} \sum\limits_{k=1}^{K} t^{0k} h_r^{ik}(x^0, b^0)} \right\} = 0, \tag{23}$$

$$g_j^i(x^0) \leq 0, \quad i \in I, \quad j = 1, 2, \cdots, m, \tag{24}$$

$$\sum_{j=1}^{m} \lambda_j^{0i} g_j^i(x^0) = 0, \quad \lambda_j^i \geq 0, \qquad i \in I, \quad j = 1, 2, \cdots, m. \tag{25}$$

The proof of the following theorem follows as in [3].

Theorem 4.1 *Assume that* $F^i(x, t, d^i, a, b)$, $g_j^i(x)$, $i \in I$, $j = 1, 2, \cdots, m$ *are convex differentiable functions on Conv S. If* $F^i(x, t, d^i, a, b)$ *and* $g_j^i(x)$ *are bounded functions for each* $x \in$ *Cov S and* $g_j^i(x)$ *satisfy CQ for* $i \in I$, *then* x^0 *is an optimal solution of GDFFNLP*(t, α, d) *if and only if there are Lagrange multipliers* $\lambda^0 \in \mathbf{R}^{p+m}$, $\lambda \geq 0$ *such that (13)-(15) are satisfied.*

Corollary 4.1 *Suppose that $F^i(x,t,d^i,a,b)$, $g_j^i(x)$, $i \in I$, $j = 1,2,\cdots,m$ are convex differentiable functions on Conv S. If $F^i(x,t,d^i,a,b)$ and $g_j^i(x)$ are bounded functions for each $x \in$ Cov S and $g_j^i(x)$ satisfy CQ for $i \in I$, then x^0 is an optimal solution of GDFFNLP(t,α,d) if and only if there are Lagrange multipliers $u^0 \geq 0$, $\lambda \geq 0$, $u \in \mathbf{R}_+^q$ and $\lambda^0 \in \mathbf{R}^{p+m}$ such that (19)-(22) are satisfied.*

The proof is follows similarly as the proof of Theorem 4.1.

Theorem 4.2 *Assume that $F^i(x,t,d^i,a,b)$ is a pseudoconvex function at $x \in$ Conv S and that $\sum\limits_{j=1}^{m} \lambda_j^i g_j^i(x)$ is a quasiconvex function. If $F^i(x,t,d^i)$ and $g_j^i(x)$ are bounded functions for each $x \in$ Conv S, and if the equations (28)-(30) are satisfied for $tin\mathbf{R}_+^k$ and $\lambda^0 \in \mathbf{R}_+^{p+m}$, then x^0 is an optimal solution of GDFFNL(t,α,d).*

Corollary 4.2 *Assume that $F^i(x,t,d^i,a,b)$ is a pseudoconvex function at $x \in$ Conv S and that $\sum\limits_{j=1}^{m} \lambda_j^i g_j^i(x)$ is a quasiconvex function. If $F^i(x,t,d^i,a,b)$ and $g_j^i(x)$ are bounded functions for each $x \in$ Conv S and there exists $u^0 \in \mathbf{R}_+^k$ and $\lambda^0 \in \mathbf{R}_+^{p+m}$ such that equations (16)-(18) are satisfied, then x^0 is an optimal solution of GDFFNLP(t,α,d).*

The proof is follows similarly as proof of Theorem 4.2.

§5. Duality Using Mond-Weir Type

According to optimality Theorems 4.1 and 4.2, we can formulate the Mond-Weir type dual (M-WDGF) of the disjunctive fractional minimax problem GDFFNLP(t,α,d) as follows:

$$\text{M}-\text{WDGF} \quad \max_{y\in\mathbf{R}^n} \sup_{i\in I} \left(H^i(y,t,\alpha,D,a,b) = \sum_{r=1}^{s}\sum_{k=1}^{K} t^k f_r^{ik}(y,a) - D^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k h_r^{ik}(y,b) \right), \tag{26}$$

where

$$D^i = \frac{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K} t^k f_r^{ik}(y,a)}{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K} t^k h_r^{ik}(y,b)} > 0, \quad i \in I.$$

Problem (M-WDGF) satisfies the following conditions:

$$\sup_{i\in I} \nabla_y \left\{ H(y,t,D,a,b) + \sum \lambda_j^i g_j^i(y) \right\} = 0, \tag{27}$$

$$\sum_{j=1}^{m} \lambda_j^i g_j^i(y) = 0, \quad \lambda_j^i \geq 0, \quad i \in I, \quad j = 1,2,\cdots,m, \tag{28}$$

$$\sum_{r=1}^{s}\sum_{k=1}^{K} t^k f_r^{ik}(y,a) - D^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k h_r^{ik}(y,b) \geq 0, \quad i \in I, \quad D^i > 0. \tag{29}$$

Theorem 5.1(Weak Duality) *Let x be feasible for GDFFNLP(t,α,d) and (u,λ,t,a,b) be feasible for (M-WDGFD). If for all feasible (y,λ,t,a,b), $H^i(y,t,\alpha,D,a,b)$ are pseudoconvex*

for each $i \in I$, *and* $\sum\limits_{j=1}^{m} \lambda_j^i g_j^i(y)$ *are quasiconvex for* $i \in I$, *then* $\inf(GDFFNLP(t,\alpha,d)) \geq \sup(M - WDGF)$.

Proof If not, then there must be that

$$\inf_{i\in I}\inf_{x\in Z}\left(\sum_{r=1}^{s}\sum_{k=1}^{K} t^k f_r^{ik}(x) - d^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k h_r^{ik}(x)\right)$$
$$< \sup_{i\in I}\sup_{y\in\mathbf{R}^n}\left(\sum_{r=1}^{s}\sum_{k=1}^{K} t^k f_r^{ik}(y) - D^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k h_r^{ik}(y)\right).$$

Hence, for $i \in I$, we get that

$$\sum_{r=1}^{s}\sum_{k=1}^{K} t^k f_r^{ik}(x) - d^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k h_r^{ik}(x) < \sum_{r=1}^{s}\sum_{k=1}^{K} t^k f_r^{ik}(y) - D^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k h_r^{ik}(y). \tag{30}$$

and by the pseudoconvexity of $H^i(y,t,D)$, (30) implies that

$$(x-y)^t \nabla_x \left(\sum_{r=1}^{s}\sum_{k=1}^{K} t^k f_r^{ik}(y) - D^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k h_r^{ik}(y)\right) < 0. \tag{31}$$

Equation (31) implies that

$$\sup_{i\in I}\sup_{y\in\mathbf{R}^n}\left((x-y)^t \nabla_x \left(\sum_{r=1}^{s}\sum_{k=1}^{K} t^k f_r^{ik}(y) - D^i \sum_{r=1}^{s}\sum_{k=1}^{K} t^k h_r^{ik}(y)\right)\right) < 0. \tag{32}$$

From equation (27) and inequality (32) it follows that

$$\sup_{i\in I}\left\{(x-y)^t \nabla_x \sum_{j=1}^{m} \mu_j^i g_j^i(y)\right\} > 0. \tag{33}$$

By (26), inequality (33) implies that

$$\sup_{i\in I}\sum_{j=1}^{m} \mu_j^i g_j^i(x) > \sup_{i\in I}\sum_{j=1}^{m} \mu_j^i g_j^i(u) > 0.$$

Then $\sum\limits_{j=1}^{m} \mu_j^i g_j^i(x) > 0$, which contradicts the assumption that x is feasible with respect to GDFFNLP(t,α,d). □

Theorem 5.2(Strong Duality) *If* x^0 *is an optimal solution of GDFFNLP*(t,α,d) *and CQ is satisfied, then there exists* $(y^0,\lambda^0,t^0,a^0,b^0) \in \mathbf{R}^{n+m}$ *is feasible for (M-WDGF) and the corresponding value of* $\inf(GDFFNLP(t,\alpha,d)) = \sup(M - WDGF)$.

Proof Since x^0 is an optimal solution of DGFFNLP(t^0,α^0,d^0) and satisfy CQ, then there is a positive integer $\lambda_j^{*i} \geq 0$, $i \in I$, $j = 1,2,\cdots,m$ such that Kuhn-Tucker conditions (27)-(29)

are satisfied. Assume that $\lambda^0 = \tau^{-1}\lambda^*$ in the Kuhn-Tucker stationary point conditions. It follows that $(y^0, \lambda^0, t^0, a^0, b^0)$ is feasible for (M-WDGF). Hence

$$\inf_{i\in I}\left(\frac{\sum\limits_{k=1}^{K} t^{0k} f_r^{ik}(x^0, a^0)}{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K} t^{0k} h_r^{ik}(x^0, b^0)}\right) = \sup_{i\in I}\left(\frac{\sum\limits_{k=1}^{K} t^{0k} f_r^{ik}(y^0, a^0)}{\sum\limits_{r=1}^{s}\sum\limits_{k=1}^{K} t^{0k} h_r^{ik}(y^0, b^0)}\right). \qquad \square$$

Theorem 5.3(Converse Duality) *Let x^0 be an optimal solution of DGFFNLP(t^0, α^0, d^0) and CQ is satisfied. If (y^*, μ^*) is an optimal solution of (M-WDFD) and $H^i(y^*, t^*, D^*)$ is strictly pseudoconvex at y^*, then $y^* = x^0$ is an optimal solution of GDFFNLP(t, α, d).*

Proof Let x^0 be an optimal solution of DGFFNLP(t^0, α^0, d^0) and CQ is satisfied. Assume that $y^* \neq x^0$. Then $(y*, \mu^*)$ is an optimal solution of (M-WDGF). Whence,

$$\inf_{i\in I}\inf_{k\in K} F^i(x^0, t^0, d^{0i}) = \sup_{i\in I}\sup_{k\in K} H^i(y^*, t^*, D^{*i}) \tag{34}$$

Because (y^*, μ^*) is feasible with respect to (M-WDGF), it follows that

$$\sum_{j=1}^{m} \mu_j^{*i} g_j^i(x^0) \leq \sum_{j=1}^{m} \mu_j^{*i} g_j^i(y^*).$$

Quasiconvexity of $\sum\limits_{j=1}^{m} \mu_j^{*i} g_j^i(x)$ implies that

$$\sup_{i\in i}(x^0 - y^*)\sum_{j=1}^{m} \nabla_x \mu_j^{*i} g_j^i(y^*) \leq 0. \tag{35}$$

From (34) and (35), it follows that

$$\sup_{i\in i}(x^0 - y^*)\nabla_y H^i(y^*, t^*, D^{*i}) \geq 0. \tag{36}$$

From (36) and the strict pseudoconvexity of at y^*, it follows that

$$\sup_{i\in i}\nabla_x F^i(x^0, t^0, d^{0i}) > \sup_{i\in i}\nabla_y H^i(y^*, t^*, D^{*i}).$$

This contradicts to (35). Hence $y^* = x^0$ is an optimal solution of DGFFNLP(t^0, α^0, d^0). $\square$

§6. **Duality Using Schaible Formula**

The Schaible dual of GDFFNLP(t, α, d) has been formulated in [27] as follows:

(SGD) $\qquad \max\limits_{(y,\mu)\in \mathbf{R}^{n+m}} D,$

where $(y, \mu) \in \mathbf{R}^n \times \mathbf{R}_+^m$ satisfying:

$$\sup_{i\in I} \nabla_x \left\{ \sum_{k=1}^{K} t^k f^{ik}(y) - D^i \sum_{k=1}^{K} t^k h^{ik}(y) + \sum_{j=1}^{m} \mu_j^i g_j^i(y) \right\} = 0, \tag{37}$$

$$\sum_{j=1}^{m} \mu_j^i g_j^i(y) \geq 0, \quad i \in I, \tag{38}$$

$$\sum_{k=1}^{K} t^k f^{ik}(y) - D^i \sum_{k=1}^{K} t^k h^{ik}(y) \geq 0, \quad i \in i, \tag{39}$$

$$D^i \geq 0 \quad \text{and} \quad \mu_j^i \geq 0, \quad i \in I, \quad j = 1, 2, \cdots, m. \tag{40}$$

Theorem 6.1(Weak Duality) *Let x be feasible with respect to GDFFNLP(t, α, d). If for all feasible (y, μ), $\sup_{i\in I} H^i(y, t, d)$ is pseudoconvex at u and $\sup_{i\in I} \sum_{j=1}^{K} \mu_j^i g_j^i(y)$ is quasiconvex, then* $\inf GDFFNLP(t, \alpha, d) \geq \sup(SGD)$.

Proof For each $i \in I$, suppose that

$$\frac{\sum_{k=1}^{K} t^k f^{ik}(y)}{\sum_{k=1}^{K} t^k h^{ik}(y)} < D^i.$$

Hence, for each $y \in \mathbf{R}^n$ and $i \in I$, we get that

$$\sum_{k=1}^{K} t^k f^{ik}(y) - D^i \sum_{k=1}^{K} t^k h^{ik}(y) < 0.$$

Therefore,

$$\sup_{i\in I} \left(\sum_{k=1}^{K} t^k f^{ik}(y) - D^i \sum_{k=1}^{K} t^k h^{ik}(y) \right) < 0. \tag{41}$$

From (39) and (41) with $t \neq 0$, we have

$$\left(\sum_{k=1}^{K} t^k f^{ik}(x) - d^i \sum_{k=1}^{K} t^k h^{ik}(x) \right) < \left(\sum_{k=1}^{K} t^k f^{ik}(y) - D^i \sum_{k=1}^{K} t^k h^{ik}(y) \right).$$

By the pseudoconvexity of $\sup_{i\in I} H^i(y, t, D)$ at u, it follows that

$$(x - y)^T \left(\sum_{k=1}^{K} t^k f^{ik}(y) - D^i \sum_{k=1}^{K} t^k h^{ik}(y) \right) < 0. \tag{42}$$

Consequently, (38) and (42) yield that

$$(x - y)^T \sum_{j=1}^{m} \mu_j^i \nabla_x g_j^i(y) > 0. \tag{43}$$

and, by the quasiconvexity of $\sum\limits_{j=1}^{m}\mu_j^i g_j^i(y)$, inequality (43) implies that

$$\sum_{j=1}^{m}\mu_j^i g_j^i(x) > \sum_{j=1}^{m}\mu_j^i g_j^i(y). \tag{44}$$

From inequalities (38) and (44) it follows that

$$\sum_{j=1}^{m}\mu_j^i g_j^i(x) > 0. \tag{45}$$

But, from the feasibility of $x \in S$ and $\mu_j^i \geq 0$, $i \in I$, $j = 1, 2, \cdots, m$, (1) implies that $\sum\limits_{j=1}^{m}\mu_j^i g_j^i(x) \leq 0$, this contradicts (45). Hence,

$$\frac{\sum\limits_{k=1}^{K} t^k f^{ik}(y)}{\sum\limits_{k=1}^{K} t^k h^{ik}(y)} \geq D^i,$$

i.e., $\inf GDFFNLP(t,\alpha,d) \geq \sup(SGD)$. □

Theorem 6.2(Strong Duality) *Let x^0 be an optimal solution of GDFFNLP(t,α,d) so that CQ is satisfied. Then there exists (y^0,μ^0) is feasible for (SDD) and the corresponding value of* $\inf GDFFNLP(t,\alpha,d) = \sup(SDD)$. *If, in addition, the hypotheses of Theorem 6.1 are satisfied, then (x^0,μ^0) is an optimal solution of (SDD).*

Proof The proof is similar to that of Theorem 5.2. □

Theorem 6.3(Converse Duality) *Suppose that $x^{)}$ is an optimal solution of GDFFNLP(t,α,d) and $g_j^i(x)$ satisfy CQ. Let the hypotheses of the above Theorem 6.1 hold. If (y^*,μ^*) is an optimal solution of (SDD) and is strictly pseudocovex at y^*, then $y^* = x^0$ is an optimal solution of DGFFNLP(t^0,α^0,d^0).*

Proof Assume that $y^* \neq x^0$, x^0 is an optimal solution DGFFNLP(t^0,α^0,d^0) and try to find a contraction. From Theorem 4.2, for each $i \in I$, it follows that

$$\frac{\sum\limits_{k=1}^{K} t^{0k} f^{ik}(x^0)}{\sum\limits_{k=1}^{K} t^{0k} h^{ik}(x^0)} = d^{0i}. \tag{46}$$

Applying (1) with (38) we get that

$$\sum_{j=1}^{m}\mu_j^{*i} g_j^i(x^0) \leq \sum_{j=1}^{m}\mu_j^{*i} g_j^i(y^*).$$

By quasiconvexity of $\sum\limits_{j=1}^{m}\mu_j^{*i} g_j^i(x)$ and for each $i \in I$, it follows that

$$(x^0 - y^*)\sum_{j=1}^{m}\nabla_x \mu_j^{*i} g_j^i(y^*) \leq 0. \tag{47}$$

From (37) and (47) it follows that

$$(x^0 - y^*)\nabla_x \left(\sum_{k=1}^{K} t^{*k} f^{ik}(y^*) - D^{*i} \sum_{k=1}^{K} t^{*k} h^{ik}(y^*) \right) \leq 0. \tag{48}$$

From (39), (48) and the strict pseudoconvexity of $\left(\sum\limits_{k=1}^{K} t^{*k} f^{ik}(y) - D^{*i} \sum\limits_{k=1}^{K} t^{*k} h^{ik}(y) \right)$ for each $i \in I$ at y^*, it follows that

$$\left(\sum_{k=1}^{K} t^{0k} f^{ik}(x^0) - d^{0i} \sum_{k=1}^{K} t^{0k} h^{ik}(x^0) \right) > \left(\sum_{k=1}^{K} t^{*k} f^{ik}(y^*) - D^{*i} \sum_{k=1}^{K} t^{*k} h^{ik}(y^*) \right). \tag{49}$$

Inequality (49) implies that

$$\left(\sum_{k=1}^{K} t^{0k} f^{ik}(x) - d^{0i} \sum_{k=1}^{K} t^{0k} h^{ik}(x) \right) > 0, \quad i \in I. \tag{50}$$

i.e., for each $i \in I$ it is follows that

$$\frac{\sum\limits_{k=1}^{K} t^{0k} f^{ik}(x)}{\sum\limits_{k=1}^{K} t^{0k} h^{ik}(x)} > d^{0i}. \tag{51}$$

Consequently,

$$\frac{\sum\limits_{k=1}^{K} t^{0k} f^{ik}(x^0)}{\sum\limits_{k=1}^{K} t^{0k} h^{ik}(x^0)} \geq \frac{\sum\limits_{k=1}^{K} t^{0k} f^{ik}(x)}{\sum\limits_{k=1}^{K} t^{0k} h^{ik}(x)} > d^{0i}, \tag{52}$$

contradicts to that (46). So that $y^* = x^0$ is an optimal solution of DGFFNLP(t^0, α^0, d^0). $\square$

§7. Conclusion

This paper addresses the solution of generalized multiobjective disjunctive programming problems, which corresponds to minmax continuous optimization problems that involve disjunctions with convex-concave nonlinear fractional objective functions. We use Dinkelbach‘s global approach for finding the maximum of this problem. We first describe the Kuhn-Tucker saddle point of nonlinear disjunctive fractional minmax programming problems by using the decision set that is the union of a family of convex sets. Also, we discuss necessary and sufficient optimality conditions for generalized nonlinear disjunctive fractional minmax programming problems. For the class of problems, we study two duals; we propose and prove weak, strong and converse duality theorems.

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Surface Embeddability of Graphs via Joint Trees

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Abstract: This paper provides a way to observe embedings of a graph on surfaces based on join trees and then characterizations of orientable and nonorientable embeddabilities of a graph with given genus.

Key Words: Surface, graph, Smarandache λ^S-drawing, embedding, joint tree.

AMS(2010): 05C15, 05C25

§1. Introduction

A drawing of a graph G on a surface S is such a drawing with no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point. A *Smarandache λ^S-drawing* of G on S is a drawing of G on S with minimal intersections λ^S. Particularly, a Smarandache 0-drawing of G on S, if existing, is called an embedding of G on S.

The term *joint three* looks firstly appeared in [1] and then in [2] in a certain detail and [3] firstly in English. However, the theoretical idea was initiated in early articles of the author [4–5] in which maximum genus of a graph in both orientable and nonorientable cases were investigated.

The central idea is to transform a problem related to embeddings of a graph on surfaces i.e., compact 2-manifolds without boundary in topology into that on polyhegons (or polygons of even size with binary boundaries). The following two principles can be seen in [3].

Principle A *Joint trees of a graph have a 1–to–1 correspondence to embeddings of the graph with the same orientability and genus i.e., on the same surfaces.*

Principle B *Associate polyhegons (as surfaces) of a graph have a 1–to–1 correspondence to joint trees of the graph with the same orientability and genus, i.e., on the same surfaces.*

The two principle above are employed in this paper as the theoretical foundation. These enable us to discuss in any way among associate polyhegons, joint trees and embeddings of a graph considered.

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§2. Layers and Exchangers

Given a surface $S = (A)$. it is divided into segments layer by layer as in the following.

The 0th layer contains only one segment, *i.e.*, $A(= A_0)$;

The 1st layer is obtained by dividing the segment A_0 into l_1 segments, *i.e.*, $S = (A_1, A_2, \cdots, A_{l_1})$, where $A_1, A_2, \cdots, A_{l_1}$ are called the 1*st layer segments*;

Suppose that on $k-1$st layer, the $k-1$st layer segments are $A_{\underline{n}_{(k-1)}}$ where $\underline{n}_{(k-1)}$ is an integral $k-1$-vector satisfied by

$$\underline{1}_{(k-1)} \leqslant (n_1, n_2, \cdots, n_{k-1}) \leqslant \underline{N}_{(k-1)}$$

with $\underline{1}_{(k-1)} = (1, 1, \cdots, 1)$, $\underline{N}_{(k-1)} = (N_1, N_2, \cdots, N_{k-1})$, $N_1 = l_1 = N_{(1)}$, $N_2 = l_{A_{N_{(1)}}}$, $N_3 = l_{A_{\underline{N}_{(2)}}}$, $\cdots$, $N_{k-1} = l_{A_{\underline{N}_{(k-2)}}}$, then the kth layer segments are obtained by dividing each $k-1$st layer segment as

$$A_{\underline{n}_{(k-1)},1}, A_{\underline{n}_{(k-1)},2}, \cdots, A_{\underline{n}_{(k-1)}, l_{A_{\underline{n}_{(k-1)}}}} \tag{1}$$

where $\underline{1}_{(k)} = (\underline{n}_{(k-1)}, 1) \leqslant (\underline{n}_{(k-1)}, i) \leqslant \underline{N}_{(k)} = (\underline{N}_{(k-1)}, N_k)$ and $N_k = l_{A_{\underline{N}_{(k-1)}}}$. Segments in (1) are called *successors* of $A_{\underline{n}_{(k-1)}}$. Conversely, $A_{\underline{n}_{(k-1)}}$ is the *predecessor* of any one in (1).

A layer segment which has only one element is called an *end segment* and others, *principle segments*. For an example, let

$$S = (1, -7, 2, -5, 3, -1, 4, -6, 5, -2, 6, 7, -3, -4).$$

Fig.2.1 shows a layer division of S and Tab.2.1, the principle segments in each layer.

For a layer division of a surface, if principle segments are dealt with vertices and edges are with the relationship between predecessor and successor, then what is obtained is a tree denoted by T. On T, by adding cotree edges as end segments, a graph $G = (V, E)$ is induced. For example, the graph induced from the layer division shown in Fig.1 is as

$$V = \{A, B, C, D, E, F, G, H, I\} \tag{2}$$

and

$$E = \{a, b, c, d, e, f, g, h, 1, 2, 3, 4, 5, 6, 7\}, \tag{3}$$

where

$$a = (A, B), b = (A, C), c = (A, D), d = (B, E),$$
$$e = (C, F), f = (C, G), g = (D, H), h = (D, I),$$

and

$$1 = (B, F), 2 = (E, H), 3 = (F, I), 4 = (G, I),$$
$$5 = (B, C), 6 = (G, H), 7 = (D, E).$$

By considering $E_T = \{a, b, c, d, e, f, g, h\}$, $\overline{E}_T = \{1, 2, 3, 4, 5, 6, 7\}$, $\delta_i = 0, i = 1, 2, \cdots, 7$, and the rotation σ implied in the layer division, a joint tree $\widehat{T}^{\delta}_{\sigma}$ is produced.

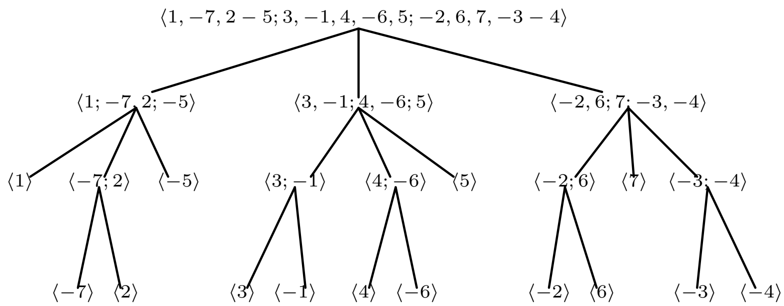

Fig.1 Layer division of surface S

Layers	Principle segments
0th layer	$A = \langle 1, -7, 2 - 5; 3, -1, 4, -6, 5; -2, 6, 7, -3 - 4\rangle$
1st layer	$B = \langle 1; -7, 2; -5\rangle, C = \langle 3, -1; 4, -6; 5\rangle,$
	$D = \langle -2, 6; 7; -3, -4\rangle$
2nd layer	$E = \langle -7; 2\rangle, F = \langle 3; -1\rangle, G = \langle 4; -6\rangle,$
	$H = \langle -2; 6\rangle, I = \langle -3; -4\rangle$

Tab.1 Layers and principle segments

Theorem 1 *A layer division of a polyhegon determines a joint tree. Conversely, a joint tree determines a layer division of its associate polyhegon.*

Proof For a layer division of a polyhegon as a polyhegon, all segments are treated as vertices and two vertices have an edge if, and only if, they are in successive layers with one as a subsegment of the other. This graph can be shown as a tree. Because of each non-end vertex with a rotation and end vertices pairwise with binary indices, this tree itself is a joint tree.

Conversely, for a joint tree, it is also seen as a layer division of the surface determined by the boundary polyhegon of the tree. □

Then, an operation on a layer division is discussed for transforming an associate polyhegon into another in order to visit all associate polyhegon without repetition.

A layer segment with all its successors is called a *branch* in the layer division. The operation of interchanging the positions of two layer segments with the same predecessor in a layer division is called an *exchanger*.

Lemma 1 *A layer division of an associate polyhegon of a graph under an exchanger is still a layer division of another associate polyhegon. Conversely, the later under the same exchanger becomes the former.*

Proof On the basis of Theorem 1, only necessary to see what happens by exchanger on a joint tree once. Because of only changing the rotation at a vertex for doing exchanger once,

exchanger transforms a joint tree into another joint tree of the same graph. This is the first conclusion. Because of exchanger inversible, the second conclusion holds. □

On the basis of this lemma, an exchanger can be seen as an operation on the set of all associate surfaces of a graph.

Lemma 2 *The exchanger is closed in the set of all associate polyhegons of a graph.*

Proof From Theorem 1, the lemma is a direct conclusion of Lemma 1. □

Lemma 3 *Let $\mathcal{A}(G)$ be the set of all associate polyhegons of a graph G, then for any S_1, $S_2 \in \mathcal{A}(G)$, there exist a sequence of exchangers on the set such that S_1 can be transformed into S_2.*

Proof Because of exchanger corresponding to transposition of two elements in a rotation at a vertex, in virtue of permutation principle that any two rotation can be transformed from one into another by transpositions, from Theorem 1 and Lemma 1, the conclusion is done. □

If $\mathcal{A}(G)$ is dealt as the vertex set and an edge as an exchanger, then what is obtained in this way is called the *associate polyhegon graph* of G, and denoted by $\mathcal{H}(G)$. From Principle A, it is also called the *surface embedding graph* of G.

Theorem 2 *In $\mathcal{H}(G)$, there is a Hamilton path. Further, for any two vertices, $\mathcal{H}(G)$ has a Hamilton path with the two vertices as ends.*

Proof Since a rotation at each vertex is a cyclic permutation(or in short a cycle) on the set of semi-edges with the vertex, an exchanger of layer segments is corresponding to a transposition on the set at a vertex.

Since any two cycles at a vertex v can be transformed from one into another by $\rho(v)$ transpositions where $\rho(v)$ is the valency of v, *i.e.*, the order of cycle(rotation), This enables us to do exchangers from the 1st layer on according to the order from left to right at one vertex to the other. Because of the finiteness, an associate polyhegon can always transformed into another by $|\mathcal{A}(G)|$ exchangers. From Theorem 1 with Principles 1–2, the conclusion is done. □

First, starting from a surface in $\mathcal{A}(G)$, by doing exchangers at each principle segments in one layer to another, a Hamilton path can always be found in considering Theorem 2 and Theorem 1. Then, a Hamilton path can be found on $\mathcal{H}(G)$.

Further, for chosen $S_1, S_2 \in \mathcal{A}(G) = V(\mathcal{H}(G))$ adjective, starting from S_1, by doing exchangers avoid S_2 except the final step, on the basis of the strongly finite recursion principle, a Hamilton path between S_1 and S_2 can be obtained. In consequence, a Hamilton circuit can be found on $\mathcal{H}(G)$.

Corollary 1 *In $\mathcal{H}(G)$, there exists a Hamilton circuit.*

Theorem 2 tells us that the problem of determining the minimum, or maximum genus of graph G has an algorithm in time linear on $\mathcal{H}(G)$.

§3. Main Theorems

For a graph G, let $\mathcal{S}(G)$ be the the associate polehegons (or surfaces) of G, and $\mathbf{S}_p$ and $\mathbf{S}_{\widetilde{q}}$, the subsets of, respectively, orientable and nonorientable polyhegons of genus $p \geqslant 0$ and $q \geqslant 1$.

Then, we have

$$\mathcal{S}(G) = \sum_{p \geqslant 0} \mathbf{S}_p + \sum_{q \geqslant 1} \mathbf{S}_{\widetilde{q}}.$$

Theorem 3 *A graph G can be embedded on an orientable surface of genus p if, and only if, $\mathcal{S}(G)$ has a polyhegon in $\mathbf{S}_p$, $p \geqslant 0$. Moreover, for an embedding of G, there exist a sequence of exchangers by which the corresponding polyhegon of the embedding can be transformed into one in $\mathbf{S}_p$.*

Proof For an embedding of G on an orientable surface of genus p, from Theorem 1 there is an associate polyhegon in $\mathbf{S}_p$, $p \geqslant 0$. This is the necessity of the first statement.

Conversely, given an associate polyhegen in $\mathbf{S}_p$, $p \geqslant 0$, from Theorems 1–2 with Principles A and B, an embedding of G on an orientable surface of genus p can be done. This is the sufficiency of the first statement.

The last statement of the theorem is directly seen from the proof of Theorem 2. □

For an orientable embedding $\mu(G)$ of G, denote by $\widetilde{\mathbf{S}}_\mu$ the set of all nonorientable associate polyhegons induced from $\mu(G)$.

Theorem 4 *A graph G can be embedded on a nonorientable surface of genus $q(\geqslant 1)$ if, and only if, $\mathcal{S}(G)$ has a polyhegon in $\widetilde{\mathbf{S}}_q$, $q \geqslant 1$. Moreover, if G has an embedding $\widetilde{\mu}$ on a nonorientable surface of genus q, then it can always be done from an orientable embedding μ arbitrarily given to another orientable embedding μ' by a sequence of exchangers such that the associate polyhegon of $\widetilde{\mu}$ is in $\widetilde{\mathbf{S}}_{\mu'}$.*

Proof For an embedding of G on a nonorientable surface of genus q, Theorem 1 and Principle B lead to that its associate polyhegon is in $\mathbf{S}_q$, $q \geqslant 1$. This is the necessity of the first statement.

Conversely, let $S_{\widetilde{q}}$ be an associate polyhegon of G in $\widetilde{\mathbf{S}}_q$, $q \geqslant 1$. From Principles A and B, an embedding of G on a nonorietable surface of genus q can be found from $S_{\widetilde{q}}$. This is the sufficiency of the first statement.

Since a nonorientable embedding of G has exactly one under orientable embedding of G by Principle A, Theorem 2 directly leads to the second statement. □

§4. Research Notes

A. Theorems 1 and 2 enable us to establish a procedure for finding all embeddings of a graph G in linear space of the size of G and in linear time of size of $\mathcal{H}(G)$. The implementation of this procedure on computers can be seen in [6].

B. In Theorems 3 and 4, it is necessary to investigate a procedure to extract a sequence of transpositions considered for the corresponding purpose efficiently.

C. On the basis of the associate polyhegons, the recognition of operations from a polyhegon of genus p to that of genus $p+k$ for given $k \geqslant 0$ have not yet be investigated. However, for the case $k=0$ the operations are just Operetions 0–2 all topological that are shown in [1–3].

D. It looks worthful to investigate the associate polyhegon graph of a graph further for accessing the determination of the maximum(orientable) and minimum(orientable or nonorientable) genus of a graph.

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Plick Graphs with Crossing Number 1

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Abstract: In this paper, we deduce a necessary and sufficient condition for graphs whose plick graphs have crossing number 1. We also obtain a necessary and sufficient condition for plick graphs to have crossing number 1 in terms of forbidden subgraphs.

Key Words: Smarandache $\mathscr{P}$-drawing, drawing, line graph, plick graph, crossing number.

AMS(2010): 05C10

§1. Introduction

All graphs considered here are finite, undirected and without loops or multiple edges. We refer the terminology of [2]. For any graph G, $L(G)$ denote the line graph of G.

A *Smarandache $\mathscr{P}$-drawing* of a graph G for a graphical property $\mathscr{P}$ is such a good drawing of G on the plane with minimal intersections for its each subgraph $H \in \mathscr{P}$. A Smarandache $\mathscr{P}$-drawing is said to be *optimal* if $\mathscr{P} = G$ and it minimizes the number of crossings. A graph is planar if it can be drawn in the plane or on the sphere in such a way that no two of its edges intersect. *The crossing number $cr(G)$* of a graph G is the least number of intersections of pairs of edges in any embedding of G in the plane. Obviously, G is planar if and only if $cr(G) = 0$. It is implicit that the edges in a drawing are Jordan arcs(hence, non-selfintersecting), and it is easy to see that a drawing with the minimum number of crossings(an optimal drawing) must be *good* drawing, that is, each two edges have at most one vertex in common, which is either a common end-vertex or a crossing. *Theta* is the result of adding a new edge to a cycle and it is denoted by θ. The corona G^+ of a graph G is obtained from G by attaching a path of length 1 to every vertex of G.

The *plick graph $P(G)$ of a graph G* is obtained from the line graph by adding a new vertex corresponding to each block of the original graph and joining this vertex to the vertices of the line graph which correspond to the edges of the block of the original graph(see[4]).

The following will be useful in the proof of our results.

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Theorem A([5]) *The line graph of a planar graph G is planar if and only if $\Delta(G) \leq 4$ and every vertex of degree 4 is a cut-vertex.*

Theorem B([3]) *Let G be a nonplanar graph. Then $cr(L(G)) = 1$ if and only if the following conditions hold:*

(1) *$cr(G) = 1$;*

(2) *$\Delta(G) \leq 4$, and every vertex of degree 4 is a cut-vertex of G;*

(3) *There exists a drawing of G in the plane with exactly one crossing in which each crossed edge is incident with a vertex of degree 2.*

Theorem C([3]) *The line graph of a planar graph G has crossing number one if and only if (1) or (2) holds:*

(1) *$\Delta(G) = 4$ and there is a unique non-cut-vertex of degree 4;*

(2) *$\Delta(G) = 5$, every vertex of degree 4 is a cut-vertex, there is a unique vertex of degree 5 and it has at most 3 incident edges in any block.*

Theorem D([4]) *The plick graph $P(G)$ of a graph G is planar if and only if G satisfies the following conditions:*

(1) *$\Delta(G) \leq 4$, and*

(2) *every block of G is either a cycle or a K_2.*

Theorem E([1]) *A graph has a planar ilne graph if and only if it has no subgraph homeomorphic to $K_{3,3}$, $K_{1,5}$, $P_4 + K_1$ or $K_2 + \overline{K}_3$.*

Remark 1([4]) For any graph, $L(G)$ is a subgraph of $P(G)$.

§2. Results

The following theorem supports the main theorem.

Theorem 1 *Let x be any edge of K_4. If G is homeomorphic to $K_4 - x$, then $cr(P(G)) = 1$.*

Proof We prove the theorem first for $G = (K_4 - x)$. One can see that the graph $P(K_4 - x)$ has 6 vertices and 13 edges. But a planar graph with 6 vertices has at most 12 edges. This shows that $P(K_4 - x)$ has crossing number at least 1. Figure 1, being drawing of $P(K_4 - x)$ concludes that $cr(P(K_4 - x)) = 1$. Suppose now G is the graph as in the statement. Referring to Figure 1, it is immediate to see that $cr(P(K_4 - x)) = 1$. □

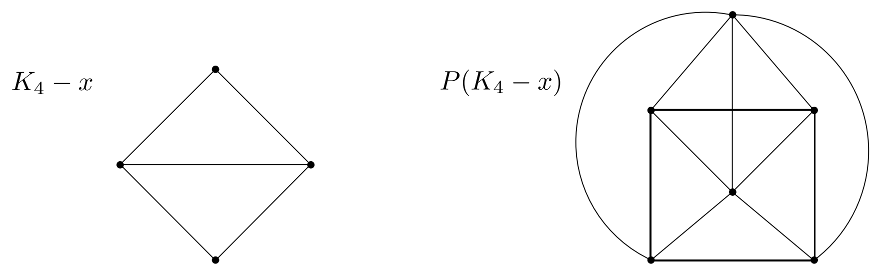

Figure 1

The following theorem gives a necessary and sufficient condition for graphs whose plick graphs have crossing number 1.

Theorem 2 *A graph G has a plick graph with crossing number 1 if and only if G is planar and one of the following holds:*

(1) $\Delta(G) = 3$, *G has exactly two non-cut-vertices of degree 3 and they are adjacent.*

(2) $\Delta(G) = 4$, *every vertex of degree 4 is a cut-vertex of G, there exists exactly one theta as a block in G such that at least one vertex of theta is a non-cut- vertex of degree 2 or 3 and every other block of G is either a cycle or a K_2.*

(3) $\Delta(G) = 5$, *G has a unique cut-vertex of degree 5 and every block of G is either a cycle or a K_2.*

Proof Suppose $P(G)$ has crossing number one. Then by Remark 1, and Theorem B, G is planar. By Theorem D, $\Delta(G) \leq 4$, then at least one block of G is neither a cycle nor a K_2.

Suppose $\Delta(G) \leq 6$. Then $K_{1,6}$ is a subgraph of G. Clearly $L(K_{1,6}) = K_6$. It is known that $cr(K_6) = 3$. By Remark 1, K_6 is a subgraph of $P(G)$ and hence $cr(P(G)) > 1$, a contradiction. This implies that $\Delta(G) \leq 5$. If $\Delta(G) \leq 2$, then $P(G)$ is planar, again a contradiction. Thus $\Delta(G) = 3$ or 4 or 5.

We now consider the following cases:

Case 1. Suppose $\Delta(G) = 3$. Then by Theorem D and since $cr(P(G)) = 1$, G has a non-cut-vertex of degree 3. Clearly G contains a subgraph homeomorphic to $K_4 - x$, so that there exist at least two non-cut-vertex of degree3. More precisely, there is an even number, say $2n$, of non-cut-vertex of degree 3. Now suppose G has at least two diagonal edges. Then there are two subcases to consider depending on whether 2 diagonal edges exist in one cycle or in two different edge disjoint cycles.

Subcase 1.1 If two diagonal edges exist in one cycle of G. Then G has a subgraph homeomorphic from K_4. The graph $P(K_4)$ has 7 vertices and 18 edges. It is known that a planar graph with 7 vertices has at most 15 edges. This shows that $P(K_4)$ must have crossing number exceeding 1 and hence $P(G)$ has crossing number greater than 1, a contradiction.

Subcase 1.2 If two diagonal edges exist in two different edge-disjoint cycles of G. Then by

Theorem 1, we see that for every subgraph of G homeomorphic to $K_4 - x$, there corresponds at least one crossing of G. Hence $P(G)$ has at least 2 crossings, a contradiction.

Hence G has exactly two non-cut-vertices of degree 3 and every other vertex of degree 3 is a cut-vertex.

Suppose a graph G has two non-cut-vertices of degree 3 and they are not adjacent. Then G contains a subgraph homeomorphic to $K_{2,3}$. On drawing $P(K_{2,3})$ in a plane one can see that $cr(P(K_{2,3})) = 2$. Since $P(K_{2,3})$ is a subgraph of $P(G)$, $P(G)$ has crossing number exceeding 1, a contradiction(see Figure 2).

Therefore, we conclude that G contains exactly two non-cut-vertices of degree 3 and these are adjacent. This proves (1).

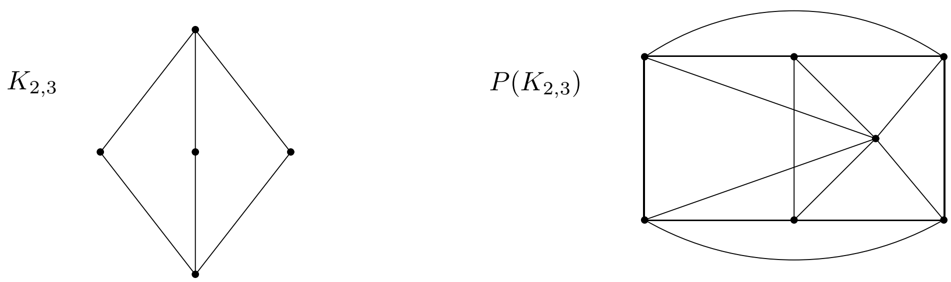

Figure 2

Case 2. Assume $\Delta(G) = 4$. We show first that every vertex of degree 4 is a cut-vertex. On the contrary suppose that G has non-cut-vertex v of degree 4. Then by Theorem C, $cr(L(G)) \geq 1$. The vertex u_1 in $P(G)$ corresponding to the block which contains a non-cut-vertex of degree 4 is adjacent to every vertex of $L(G)$. We obtain the drawing of $P(G)$ with 3 crossings.

Assume now G has at least two blocks each of which is a theta. By Theorem 1 and case 1 of this theorem, we see that for every subgraph of G homeomorphic to $K_4 - x$, there correspond to at least 2 crossings of G, a contradiction.

Suppose there exists exactly one theta S as a block in G such that none of its vertices is a non-cut-vertex of degree 2 or 3. Assume all vertices of theta S have degree 4 in G. Then by Theorem A, $L(S)$ is planar. Let v_1 be the vertex of $L(G)$ corresponding to the chord of a cycle C of theta. The vertex w_1 in $P(G)$ corresponding to the block theta S is adjacent to every vertex of $L(C)$ without crossings. In $P(G) - v_1w_1$, the vertex w_1 is adjacent to every vertex of $L(S) - v_1$ without crossings. By the definition of $P(G)$, the vertices v_1 and w_1 are adjacent in $P(G)$. The edge v_1w_1 crosses at least two edges of $L(G)$. On drawing of $P(G)$ in the plane, it has at least two crossings, a contradiction. This proves that $\Delta(G) = 4$, there exist exactly one theta as a block in G such that at least one vertex of theta is either a non-cut-vertex of degree 2 or 3.

Suppose every block of G different from theta block is neither a cycle nor a K_2. It implies that G has a block which is a subgraph homeomorphic to $K_4 - x$. By Cases 1 and 2 of this theorem, we see that for every subgraph of G homeomorphic to $K_4 - x$, there corresponds at

least one crossing of G. Hence $P(G)$ has at least 2 crossings, a contradiction.

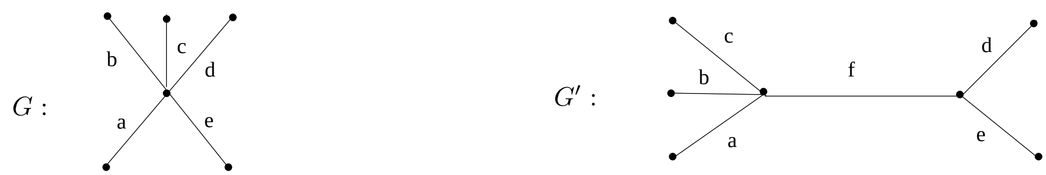

Figure 3

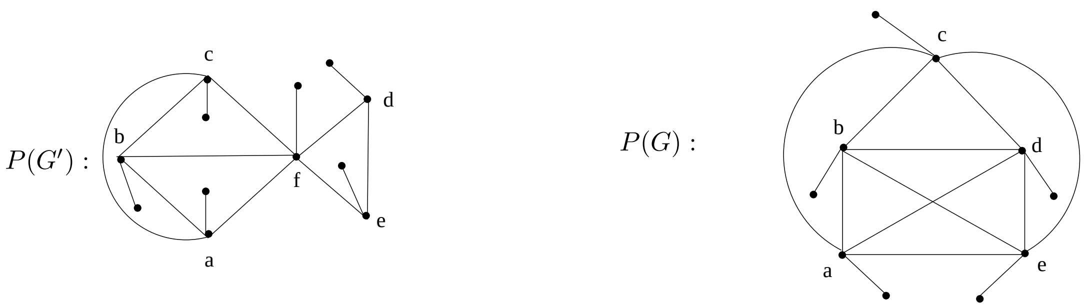

Figure 4

Case 3. Assume $\Delta(G) = 5$. Suppose G has at least two vertices of degree 5. Then by Theorem C, $L(G)$ has crossing number at least 2. By Remark 1, $cr(P(G)) \geq 2$, which is a contradiction. Thus G has a unique vertex of degree 5.

Suppose G has a vertex v of degree 5 and at least one block of G is neither a cycle nor a K_2. Then some block of G has a subgraph homeomorphic to $K_4 - x$. By Case 1 of this theorem $cr(P(K_4 - x)) \geq 1$ and the 5 edges incident to v form K_5 as a subgraph in $P(G)$. Hence $cr(P(K_4 - x)) \geq 2$, a contradiction.

Conversely, suppose G is a planar graph satisfying (1) or (2) or (3). Then by Theorem D, $P(G)$ has crossing number at least 1. We now show that its crossing number is at most 1. First suppose (1) holds. Then G has exactly one block, say H, homeomorphic to $K_4 - x$ which contains 2 adjacent non-cut-vertices of degree 3. By Theorem 1, $cr(P(H)) = 1$. By Theorem D, all other remaining blocks of G have a planar plick graph. Hence $P(G)$ has crossing number 1.

Assume (2) holds. Let u be a cut-vertex of degree 4. The vertex u has a non-cut-vertex of degree 3 in a block for otherwise, G does contain a subgraph homeomorphic to $K_4 - x$ which is impossible. By virtue of Theorem 1, for a non-cut-vertex of G of degree 3, there corresponds one crossing in $P(G)$. However $P(G)$ can not have more than one crossing since the removal of any edge in a block containing u, yields a graph H such that $P(H)$ is planar by Theorem D. It follows easily that $P(G)$ has crossing number 1.

Suppose (3) holds. The edges at the vertex v of the degree 5 can be split into sets of sizes

2 and 3 so that no edges in different sets are in the same block. Transform G to G' as in Figure 3. Then $P(G')$ is again planar. Thus $P(G)$ can be drawn with only one crossing as shown in Figure 4. $\square$

§3. Forbidden Subgraphs

By using Theorem 2, we now characterize graphs whose plick graphs have crossing number 1 in terms of forbidden subgraphs.

Theorem 3 *The plick graph of a connected graph G has crossing number 1 if and only if G has no subgraphs homeomorphic from any one of the graphs of Figure 5 or G has subgraph θ^+ such that none of the vertices of theta have non-cut-vertices of degree 2 or 3.*

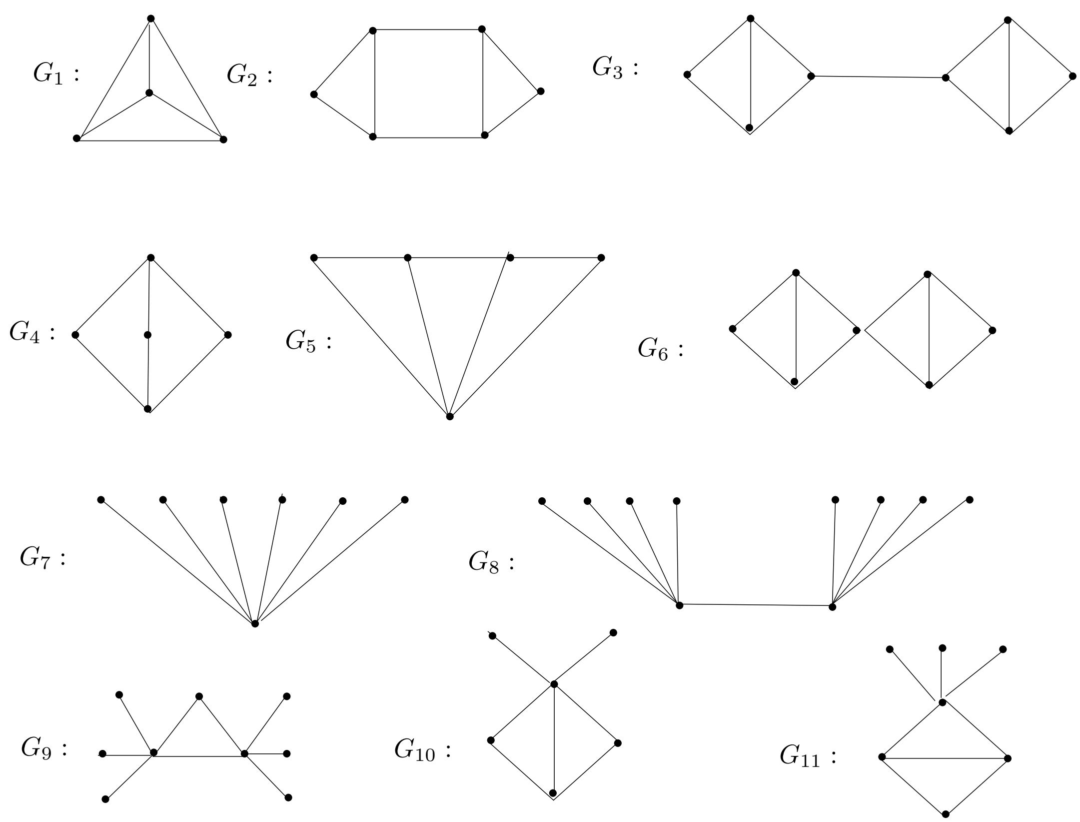

Figure 5

Proof Suppose G has a plick graph with crossing number one. We now show that all graphs homeomorphic from any one of the graphs of Figure 5 or a subgraph θ^+ such that none of the vertices of theta have non-cut-vertices of degree 2 or 3, have no plick graph with crossing number one. This result follows from Theorem 2, since graphs homeomorphic from G_1, G_2 or G_3 have more than two non-cut-vertices of degree three. Graphs homeomorphic from G_4 have two non-cut-vertices of degree 3 which are not adjacent. Graphs homeomorphic from

G_5 have a vertex of degree 4 which is a non-cut-vertex. Graphs homeomorphic from G_6 have more than one theta. θ^+ has exactly one block which is a theta and none of its vertices have non-cut-vertices of degree 2 or 3. Graphs homeomorphic from G_7 have $\Delta(G_7) > 5$. Graphs homeomorphic from G_8 or G_9 have two or more vertices of degree 5. Graphs homeomorphic to G_{10} or G_{11} have a block which is neither a cycle nor a K_2.

Conversely, suppose G is a graph which does not contain a subgraph homeomorphic from any one of the graphs of Figure 5 or G has exactly one subgraph theta as a block such that none of the vertices of theta have non-cut-vertices of degree 2 or 3. First we prove condition (1) of Theorem 2. Suppose G contains more than two non-cut-vertices of degree 3. Then it is easy to see that G is a planar graph with at least 2 diagonal edges. Now consider 2 cases depending on whether the 2 diagonal edges exist in one block or in two different blocks.

Case 1. Suppose two diagonal edges exist in one block of G, then G has a subgraph homeomorphic from G_1 or G_2.

Case 2. Suppose two diagonal edges exist in two different blocks of G, then G has a subgraph homeomorphic from G_3.

In each case we have a contradiction. Hence G has at most two non-cut-vertices of degree 3. Suppose G has exactly two nonadjacent non-cut-vertices of degree 3. Then there exist 3 disjoint paths between these two non-cut-vertices of degree 3. Clearly G contains a subgraph homeomorphic from G_4, a contradiction. Thus G has exactly two adjacent non-cut-vertices of degree 3.

Since G does not contain a subgraph homeomorphic from G_7 i.e, $K_{1,6}$, $\Delta(G) \leq 5$. Also since $\Delta(G) \geq 4$, if it follows that $\Delta(G) = 4$ or 5.

Suppose G has a vertex v of degree 4. We prove that v is a cut-vertex. If not, let a, b, c and d be the vertices of G adjacent to v. Then there exist paths between every pair of vertices of a, b, c and d not containing v. Then it is proved in Theorem E, G has a subgraph homeomorphic from G_5, this is a contradiction. Thus v is a cut-vertex and every vertex of degree 4 is a cut-vertex.

Suppose that a cut-vertex of degree 4 lies on two blocks, each of which is a theta. Then G has a subgraph homeomorphic from G_6. This is a contradiction. G has exactly one block which is a theta such that at least one vertex of theta is either a non-cut-vertex of degree 2 or 3, for otherwise a forbidden subgraph has exactly one theta as a block such that none of the vertices of theta have non-cut-vertices of degree 2 or 3 would appear in G.

Suppose G has two vertices v_1 and v_2 of degree 5. Since G is a connected, v_1 and v_2 are connected by a path P and let (v_1, a_i) and (v_2, b_j), $i, j = 1, 2, 3, 4$, be edges of G. We consider the following possibilities.

If $a_i \neq b_j$ for $i, j = 1, 2, 3, 4$, then G contains a subgraph homeomorphic from G_8, a contradiction.

If there exists a path between a vertex of a_i and a vertex of b_j, then G has a subgraph homeomorphic from G_9, a contradiction.

If $a_i = b_j$, for $i, j = 1, 2$, then clearly G contains a subgraph homeomorphic from G_{10}, a contradiction.

This proves that G has exactly one vertex v of degree 5.

Suppose G has a vertex v of degree 5. We show that v is a cut-vertex. If possible let us assume that G has a non-cut-vertex of degree 5. In this case Greenwell and Hemminger showed in [1] that G must contain a subgraph homeomorphic from G_5, a contradiction.

Suppose G has a unique cut-vertex v of degree 5 and it lies on blocks, one block which is neither a cycle nor a K_2. Then G contains a subgraph homeomorphic from G_{10} or G_{11}.

Thus Theorem 2 implies that G has a plick graph with crossing number one. $\square$

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Effects of Foldings on Free Product of Fundamental Groups

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Abstract: In this paper, we will introduce free fundamental groups of some types of manifolds. Some types of conditional foldings restricted on the elements on free group and their fundamental groups are deduced. Also, the fundamental group of the limit of foldings on a wedge sum of two manifolds is obtained. Theorems governing these relations will be achieved.

Key Words: Manifolds, Folding, fundamental group, Free group

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§1. Introduction

In this article the concept of foldings will be discussed from viewpoint of algebra. The effect of foldings on the manifold M or on a finite number of product manifolds $M_1xM_2x...xM_n$ on the fundamental group $\pi_1(M)$ and $\pi_1(M_1xM_2x\cdots xM_n)$ will be presented. The folding of a manifold was, firstly introduced by Robertson 1977 [14]. More studies on the folding of many types of manifolds were studied in [2-4 and 6-9]. The unfolding of a manifold introduced in [5]. Some application of the folding of a manifold discussed in [1]. The fundamental groups of some types of a manifold are discussed in [10-13].

§2. Definitions

1. The set of homotopy classes of loops based at the point $x_\circ$ with the product operation $[f][g] = [f \cdot g]$ is called the fundamental group and denoted by $\pi_1(X, x_\circ)$ [11].

2. Let M and N be two smooth manifolds of dimension m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if for every piecewise geodesic path $\gamma : I \to M$ the induced path $f \circ \gamma : I \to N$ is piecewise geodesic and of the same length as γ [14]. If f does not preserve length it is called topological folding [9].

3. Let M and N be two smooth manifolds of the same dimension. A map $g : M \to N$ is said to be unfolding of M into N if every piecewise geodesic path $\gamma : I \to M$, the induced path $g \circ \gamma : I \to N$ is piecewise geodesic with length greater than γ [5].

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4. Given spaces X and Y with chosen points $x_o \in X$ and $y_o \in Y$, then the wedge sum $X \vee Y$ is the quotient of the disjoint union $X \cup Y$ obtained identifying x_o and y_o to a single point [11].

5. Let S be an arbitrary set. A free group on the set S is a group F together with a function $\phi : S \to F$ such that the following condition holds: For any function $\psi : S \to H$, there exist a unique homomorphism $f : F \to H$ such that $f \circ \phi = \psi$ [12].

§3. Main Results

Paving the stage to this paper, we then introduce the following

(1) $\pi_1(T) = \{([\alpha_1]^k, [\beta_1]^m), ([\alpha_2]^k, [\beta_2]^m), ..., ([\alpha_n]^k, [\beta_n]^m); [\alpha_i], [\beta_i] \in \pi_1(S^1), k, m \in Z, k \neq 0, m \neq 0, i = 1, 2, ..., n\}$

(2) $\pi_1(T)\mathrm{mod}(k, m) = \{([\alpha_1], [\beta_1]), ([\alpha_2], [\beta_2]), ..., ([\alpha_n], [\beta_n]) : [\alpha_i]^k = 1, [\beta_i]^m = 1\ [\alpha_i], [\beta_i] \in \pi_1(S^1), k, m \in Z^+, k \neq 0, m \neq 0, i = 1, 2, ..., n\}$.

Where, $\pi_1(S^1)$ is a fundamental group of the circle ,T is the torus $[\alpha]^n = \underbrace{[\alpha] \times [\alpha] \times .... \times [\alpha]}_{n-terms}$, and $T^n = \underbrace{T \times T \times .... \times T}_{n-terms}$.

Let $\pi_1(S_1^1)$, $\pi_1(S_2^1)$ be two fundamental groups. Then the free product of $\pi_1(S_1^1)$, $\pi_1(S_2^1)$ is the group $\pi_1(S_1^1)* \pi_1(S_2^1)$consisting of all reduced words $a_1a_2a_3....a_m$ of an arbitrary finite length $m \geq 0$ such that $a_i \in \pi_1(S_1^1)$ or $a_i \in \pi_1(S_2^1), i = 1, 2, ..., m$, then we can represent the elements a_i as of the forms $a_i = [\alpha]^{n_i}$ or $a_i = [\beta]^{n_i}$ where $n_i \in Z, n_i \neq 0$ and α, β are two loops that goes once a round S_1^1, S_2^1 respectively. Also, if $F : S_1^1 \vee S_2^1 \longrightarrow S_1^1 \vee S_2^1$ is a folding, then the induced folding $\overline{F} :\pi_1(S_1^1) * \pi_1(S_2^1) \longrightarrow \pi_1(S_1^1) * \pi_1(S_2^1)$ has the following forms:

$$\overline{F}(\pi_1(S_1^1) * \pi_1(S_2^1)) = \overline{F}(\pi_1(S_1^1)) * \pi_1(S_2^1),$$
$$\overline{F}(\pi_1(S_1^1) * \pi_1(S_2^1)) = \pi_1(S_1^1) * \overline{F}(\pi_1(S_2^1)),$$
$$\overline{F}(\pi_1(S_1^1) * \pi_1(S_2^1)) = \overline{F}(\pi_1(S_1^1)) * \overline{F}(\pi_1(S_2^1)).$$

Theorem 3.1 *If $F_i : S_1^1 \vee S_2^1 \longrightarrow S_1^1 \vee S_2^1$, $i = 1, 2$ are two types of of foldings, where $F_I(S_j^1) =.S_j^1$, $j = 1, 2$, then there are induced foldings $\overline{F_i} :\pi_1(S_1^1) * \pi_1(S_2^1) \longrightarrow \pi_1(S_1^1) * \pi_1(S_2^1)$ such that $\overline{F_i}(\pi_1(S_1^1))*\pi_1(S_2^1) \approx Z$.*

Proof First, let $F_1 : S_1^1 \vee S_2^1 \longrightarrow S_1^1 \vee S_2^1$ is folding such that $F_1(S_1^1) = S_1^1$, $F_1(S_2^1) = S_1^1$ as in Fig.1. Then we can express each element $g = a_1a_2a_3....a_m$, $m \geq 1$ of $\pi_1(S_1^1) * \pi_1(S_2^1)$ in the following forms

$[\alpha]^{n_1}\,[\beta]^{n_2}\,[\alpha]^{n_3}\, ...\, [\alpha]^{n_{m-1}}\,[\beta]^{n_m}$, $[\alpha]^{n_1}\,[\beta]^{n_2}\,[\alpha]^{n_3} \cdots [\beta]^{n_{m-1}}\,[\alpha]^{n_m}$,

$[\beta]^{n_1}\,[\alpha]^{n_2}\,[\beta]^{n_3}\, ...\, [\beta]^{n_{m-1}}\,[\alpha]^{n_m}$, or $[\beta]^{n_1}\,[\alpha]^{n_2}\,[\beta]^{n_3}\, ...\, [\alpha]^{n_{m-1}}\,[\beta]^{n_m}$,

where $n_1, n_2, \cdots, n_m$ are nonzero integers and $[\alpha]^{n_k} \in \pi_1(S_1^1), [\beta]^{n_k} \in \pi_1(S_2^1), k = 1, 2, ..m$.

Then, the induced folding of the element g is

$$\overline{F_1}(g) = \overline{F_1}([\alpha]^{n_1})\overline{F_1}([\beta]^{n_2})\overline{F_1}([\alpha]^{n_3})\cdots\overline{F_1}([\alpha]^{n_{m-1}})\overline{F_1}([\beta]^{n_m}),$$
$$\overline{F_1}([\alpha]^{n_1})\overline{F_1}([\beta]^{n_2})\overline{F_1}([\alpha]^{n_3})\cdots\overline{F_1}([\beta]^{n_{m-1}})\overline{F_1}([\alpha]^{n_m}),$$
$$\overline{F_1}([\beta]^{n_1})\overline{F_1}([\alpha]^{n_2})\overline{F_1}([\beta]^{n_3})\cdots\overline{F_1}([\beta]^{n_{m-1}})\overline{F_1}([\alpha]^{n_m}),$$
$$\overline{F_1}([\beta]^{n_1})\overline{F_1}([\alpha]^{n_2})\overline{F_1}([\beta]^{n_3})....\overline{F_1}([\alpha]^{n_{m-1}})\overline{F_1}([\beta]^{n_m}).$$

Since $\overline{F_1}([\alpha]^{n_k}) = [\alpha]^{n_k}$, $\overline{F_1}([\beta]^{n_k}) = [\alpha]^{n_k}$ it follows that $\overline{F_1}(a_1a_2a_3...a) = [\alpha]^{(n_1+n_2+\cdots+n_m)}$. Hence, there is an induced folding $\overline{F_i}:\pi_1(S^1_1)*\pi_1(S^1_2) \longrightarrow \pi_1(S^1_1)*\pi_1(S^1_2)$ such that $\overline{F_i}(\pi_1(S^1_1)*\pi_1(S^1_2)) = \pi_1(S^1_1)$, and so $\overline{F_i}(\pi_1(S^1_1)*\pi_1(S^1_2)) \approx Z$. Similarly , if $F_2 : S^1_1 \vee S^1_2 \longrightarrow S^1_1 \vee S^1_2$ is folding, such that $F_2(S^1_1) = S^1_2$, $F_2(S^1_2) = S^1_2$, then there is an induced folding $\overline{F_2}:\pi_1(S^1_1)*\pi_1(S^1_2) \longrightarrow \pi_1(S^1_1)*\pi_1(S^1_2)$ such that $\overline{F_2}(\pi_1(S^1_1)*\pi_1(S^1_2)) \approx Z$.

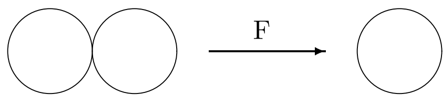

Fig.1

Theorem 3.2 *If $F_i : S^1_1 \vee S^1_2 \longrightarrow S^1_1 \vee S^1_2, i = 1,2$ are two types of foldings such that $F_i(S^1_j) = S^1_i, j = 1,2$. Then,$\pi_1(\lim_{n\to\infty} F_{i_n}(S^1_1 \vee S^1_2))$ is isomorphic to Z.*

Proof Let $F_i(S^1_j) = S^1_i$ then $\lim_{n\to\infty} F_{i_n}(S^1_1 \vee S^1_2) = S^1_i$ as in Fig.2. Thus, $\pi_1(\lim_{n\to\infty} F_{i_n}(S^1_1 \vee S^1_2))$ $=S^1_i$, Therefore $\pi_1(\lim_{n\to\infty} F_{i_n}(S^1_1 \vee S^1_2))$ is isomorphic to Z. □

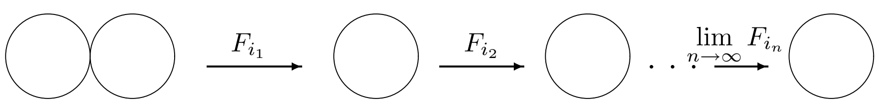

Fig.2

Theorem 3.3 *Let $F : S^1_1 \vee S^1_2 \longrightarrow S^1_1 \vee S^1_2$ be a folding, where $F(S^1_i) \neq S^1_i, i = 1,2.$Then there is an induced folding $\overline{F} : \pi_1(S^1_1)*\pi_1(S^1_2) \longrightarrow \pi_1(S^1_1)*\pi_1(S^1_2)$such that $\overline{F}\pi_1(S^1_1)*\pi_1(S^1_2)) = 0$.*

Proof Let $F : S^1_1 \vee S^1_2 \longrightarrow S^1_1 \vee S^1_2$ be a folding such that $F(S^1_1) \neq S^1_1, F(S^1_i) \neq S^1_i$ as in Fig. (3) .Then, we can express each element $g = a_1a_2a_3....a_m$, $m \geq 1$ of $\pi_1(S^1_1)*\pi_1(S^1_2)$ in the following forms:

$$[\alpha]^{n_1}[\beta]^{n_2}[\alpha]^{n_3}\cdots[\alpha]^{n_{m-1}}[\beta]^{n_m}, \quad [\alpha]^{n_1}[\beta]^{n_2}[\alpha]^{n_3}\cdots[\beta]^{n_{m-1}}[\alpha]^{n_m},$$
$$[\beta]^{n_1}[\alpha]^{n_2}[\beta]^{n_3}\cdots[\beta]^{n_{m-1}}[\alpha]^{n_m}, \quad [\beta]^{n_1}[\alpha]^{n_2}[\beta]^{n_3}\cdots[\alpha]^{n_{m-1}}[\beta]^{n_m},$$

where $n_1, n_2, \cdots, n_m$ are nonzero integers and $[\alpha]^{n_k} \in \pi_1(S^1_1), [\beta]^{n_k} \in \pi_1(S^1_2), k = 1,2,\cdots,m$.

Then the induced folding of the element g is

$$\begin{aligned}
&\overline{F_1}(g) = \overline{F_1}([\alpha]^{n_1})\overline{F_1}([\beta]^{n_2})\overline{F_1}([\alpha]^{n_3})\cdots\overline{F_1}([\alpha]^{n_{m-1}})\overline{F_1}([\beta]^{n_m})\\
&= [\alpha]^{n_1}\,[\beta]^{n_2}\,[\alpha]^{n_3}\cdots[\alpha]^{n_{m-1}}\,[\beta]^{n_m}\,,\\
&\overline{F_1}([\alpha]^{n_1})\overline{F_1}([\beta]^{n_2})\overline{F_1}([\alpha]^{n_3})\cdots\overline{F_1}([\beta]^{n_{m-1}})\overline{F_1}([\alpha]^{n_m})\\
&= [\alpha]^{n_1}\,[\beta]^{n_2}\,[\alpha]^{n_3}\cdots[\beta]^{n_{m-1}}\,[\alpha]^{n_m}\,,\\
&\overline{F_1}([\beta]^{n_1})\overline{F_1}([\alpha]^{n_2})\overline{F_1}([\beta]^{n_3})\cdots\overline{F_1}([\beta]^{n_{m-1}})\overline{F_1}([\alpha]^{n_m})\\
&= [\beta]^{n_1}\,[\alpha]^{n_2}\,[\beta]^{n_3}\cdots[\beta]^{n_{m-1}}\,[\alpha]^{n_m}\,,\\
&\overline{F_1}([\beta]^{n_1})\overline{F_1}([\alpha]^{n_2})\overline{F_1}([\beta]^{n_3})\cdots\overline{F_1}([\alpha]^{n_{m-1}})\overline{F_1}([\beta]^{n_m})\\
&= [\beta]^{n_1}\,[\alpha]^{n_2}\,[\beta]^{n_3}\cdots[\alpha]^{n_{m-1}}\,[\beta]^{n_m}\,.
\end{aligned}$$

It follows from $\left[\overset{\curlywedge}{\alpha}\right],\left[\overset{\curlywedge}{\beta}\right] \longrightarrow$ identity element, that there is an induced folding $\overline{F}:\pi_1(S_1^1) * \pi_1(S_2^1) \longrightarrow \pi_1(S_1^1) * \pi_1(S_2^1)$such that $\overline{F}(\pi_1(S_1^1) * \pi_1(S_2^1)) = 0$.

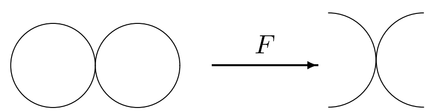

Fig.3

Corollary 1 *If* $F_i : S_1^1 \vee S_2^1 \longrightarrow S_1^1 \vee S_2^1, i = 1, 2$ *are two types of foldings such that*

$$F_i(S_i^1) = S_i^1, F_j(S_i^1) \neq S_i^1, j = 1, 2, i \neq j.$$

Then there are induced foldings $\overline{F_i}:\pi_1(S_1^1) * \pi_1(S_2^1) \longrightarrow \pi_1(S_1^1) * \pi_1(S_2^1)$*such that* $\overline{F_i}(\pi_1(S_1^1) * \pi_1(S_2^1)) \approx Z$.

Theorem 4 *If* $F : S_1^1 \vee S_2^1 \longrightarrow S_1^1 \vee S_2^1$ *is a folding such that* $F(S_i^1) \neq S_i^1, i = 1, 2$. *Then,*

$$\pi_1(\lim_{n\to\infty} F_n(S_1^1 \vee S_2^1))$$

is the identity group.

Proof If $F(S_i^1) \neq S_i^1$, $i = 1, 2$ then $\lim\limits_{n\to\infty} F_n(S_1^1 \vee S_2^1)$ is a point as in Fig.4, and so $\pi_1(\lim\limits_{n\to\infty} F_n(S_1^1 \vee S_2^1))$ is the fundamental group of a point. Therefore, we get that $\pi_1(\lim\limits_{n\to\infty} F_n(S_1^1 \vee S_2^1)) = 0$. □

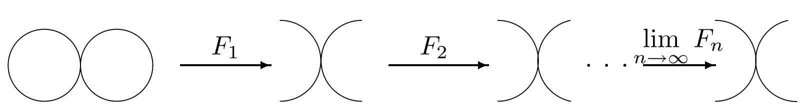

Fig.4

Theorem 5 *If* $F_i : S_1^1 \vee S_2^1 \longrightarrow S_1^1 \vee S_2^1, i = 1, 2$ *are two types of foldings such that* $F_i(S_i^1) = S_i^1, F_j(S_i^1) \neq S_i^1, j = 1, 2, i \neq j$. *Then* $\pi_1(\lim\limits_{n\to\infty} F_{i_n}(S_1^1 \vee S_2^1))$ *is isomorphic to* Z.

Proof It follows from $F_i(S_i^1) = S_i^1, F_j(S_i^1) \neq S_i^1, j = 1, 2, i \neq j$.that the limit of one circle is a circle and the limit of the other circle is a point, so $\lim_{n\to\infty} F_n(S_1^1 \vee S_2^1)) = S_i^1$ as in Fig.5. Thus, $\pi_1(\lim_{n\to\infty} F_{i_n}(S_1^1 \vee S_2^1)) = \pi_1(S_i^1)$. Therefore $\pi_1(\lim_{n\to\infty} F_{i_n}(S_1^1 \vee S_2^1))$ is isomorphic to Z. □

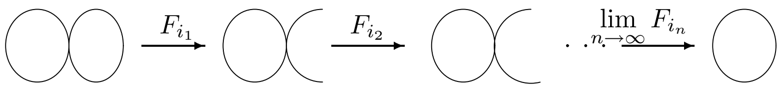

Fig.5

Now, we will generalize the above concepts for the tours Consider $\pi_1(T_1^1)$,$\pi_1(T_2^1)$, are two fundamental groups. Then, the free product of $\pi_1(T_1^1)$,$\pi_1(T_2^1)$, is the group $\pi_1(T_1^1) * \pi_1(T_2^1)$ consisting of all reduced words of $a_1a_2a_3....a_m$ of an arbitrary finite length $m \geq 0$ such that

$a_i \in \pi_1(T_1^1)$ or $a_i \in \pi_1(T_2^1)$ and so, we can represent the elements a_i as of the forms $a_i = ([\alpha_1]^{n_i}, [\beta_1]^{k_i})$ or $a_i = ([\alpha_2]^{n_i}, [\beta_2]^{k_i})$ where $n_i, k_i \in Z, n_i \neq 0$, $k_i \neq 0$ where $([\alpha_1]^{n_i}, [\beta_1]^{k_i}) \in \pi_1(T_1^1)$, $([\alpha_2]^{n_i}, [\beta_2]^{k_i}) \in \pi_1(T_2^1)$and α_j, β_j are loops that goes once a round the generators of T_j for $j = 1, 2$.Then if $F : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1$is a folding, then the induced folding $\overline{F}$:$\pi_1(T_1^1) * \pi_1(T_2^1) \longrightarrow \pi_1(T_1^1) * \pi_1(T_2^1)$ has the following forms:

$$\overline{F}(\pi_1(T_1^1) * \pi_1(T_2^1)) = \overline{F}(\pi_1(T_1^1)) * \pi_1(T_2^1),$$
$$\overline{F}(\pi_1(T_1^1) * \pi_1(T_2^1)) = \pi_1(T_1^1) * \overline{F}(\pi_1(T_2^1)),$$
$$\overline{F}(\pi_1(T_1^1) * \pi_1(T_2^1)) = \overline{F}(\pi_1(T_1^1)) * \overline{F}(\pi_1(T_2^1)).$$

Theorem 6 *If* $F_i : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1, i = 1, 2$ *are two types of foldings, where* $F_i(T_j^1) = T_i, j = 1, 2$. *Then, there are induced foldings* $\overline{F_i}$:$\pi_1(T_1^1) * \pi_1(T_2^1) \longrightarrow \pi_1(T_1^1) * \pi_1(T_2^1)$ *such that* $\overline{F_i}(\pi_1(T_1^1) * \pi_1(T_2^1)) \approx Z \times Z$.

Proof First, if $F_1 : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1$ is a folding such that $F_1(T_1^1) = T_1, F_1(T_2^1) = T_1$ as in Fig.6. Then we can express each element $g = a_1a_2...a_m, m \geq 1$ of $\pi_1(T_1^1) * \pi_1(T_2^1)$ in the following forms.

$$([\alpha_1]^{n_1}, [\beta_1]^{k_1})([\alpha_2]^{n_2}, [\beta_2]^{k_2})([\alpha_1]^{n_3}, [\beta_1]^{k_3}) \cdots ([\alpha_1]^{n_{m-1}}, [\beta_1]^{k_{m-1}})([\alpha_2]^{n_m} [\beta_2]^{k_m}),$$
$$([\alpha_1]^{n_1}, [\beta_1]^{k_1})([\alpha_2]^{n_2}, [\beta_2]^{k_2})([\alpha_1]^{n_3}, [\beta_1]^{k_3}) \cdots ([\alpha_2]^{n_{m-1}}, [\beta_2]^{k_{m-1}})([\alpha_1]^{n_m} [\beta_1]^{k_m}),$$
$$([\alpha_2]^{n_1}, [\beta_2]^{k_1})([\alpha_1]^{n_2}, [\beta_1]^{k_2})([\alpha_2]^{n_3}, [\beta_2]^{k_3}) \cdots ([\alpha_2]^{n_{m-1}}, [\beta_2]^{k_{m-1}})([\alpha_1]^{n_m} [\beta_1]^{k_m}),$$
$$([\alpha_2]^{n_1}, [\beta_2]^{k_1})([\alpha_1]^{n_2}, [\beta_1]^{k_2})([\alpha_2]^{n_3}, [\beta_2]^{k_3}) \cdots ([\alpha_1]^{n_{m-1}}, [\beta_1]^{k_{m-1}})([\alpha_1]^{n_m} [\beta_1]^{k_m}),$$

where $n_1, n_2, \cdots, n_m$, $k_1, k_2, \cdots, k_m$ are nonzero integers,

$$([\alpha_i]^{n_1}, [\beta_i]^{k_1}) \in \pi_1(T_1^1), ([\alpha_i]^{n_2}, [\beta_i]^{k_2}) \in \pi_1(T_2^1).$$

Since $\overline{F_1}([\alpha_1]^{n_1}, [\beta_1]^{k_1}) = ([\alpha_1]^{n_1}, [\beta_1]^{k_1}), \overline{F_1}([\alpha_2]^{n_1}, [\beta_2]^{k_1}) = ([\alpha_1]^{n_1}, [\beta_1]^{k_1})$, it follows that there is an induced folding $\overline{F_i}$:$\pi_1(T_1^1) * \pi_1(T_2^1) \longrightarrow \pi_1(T_1^1) * \pi_1(T_2^1)$ such that $\overline{F_1}(\pi_1(T_1^1) * \pi_1(T_2^1)) = \pi_1(T_1^1)$, and so $\overline{F_1}(\pi_1(T_1^1) * \pi_1(T_2^1)) \approx Z \times Z$. Similarly , if $F_2 : T_1^1 \vee T_2^1 \longrightarrow$

$T_1^1 \vee T_2^1$ is folding, such that $F_2(T_1^1) = T_1, F_2(T_2^1) = T_1$, then there is an induced folding $\overline{F_2}(\pi_1(T_1^1) * \pi_1(T_2^1)) = \pi_1(T_1^1)$ such that $\overline{F_2}(\pi_1(T_1^1) * \pi_1(T_2^1)) \approx Z \times Z$. □

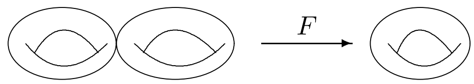

Fig.6

Theorem 7 *If $F_i : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1, i = 1, 2$ are two types of foldings, where $F_i(T_j^1) = T_i, j = 1, 2$. Then $\pi_1(\lim\limits_{n\to\infty} F_{i_n}(T_1^1 \vee T_2^1)) \approx Z \times Z$.*

Proof If $F_i : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1, i = 1, 2$ are two types of foldings, where $F_i(T_j^1) = T_i, j = 1, 2$, then $\lim\limits_{n\to\infty} F_{i_n}(T_1^1 \vee T_2^1) = T_i^1$ as in Fig.7. Thus $\pi_1(\lim\limits_{n\to\infty} F_{i_n}(T_1^1 \vee T_2^1)) = \pi_1(T_i^1)$,since $\pi_1(T_i^1) \approx Z \times Z$ we have $\pi_1(\lim\limits_{n\to\infty} F_{i_n}(T_1^1 \vee T_2^1)) \approx Z \times Z$. □

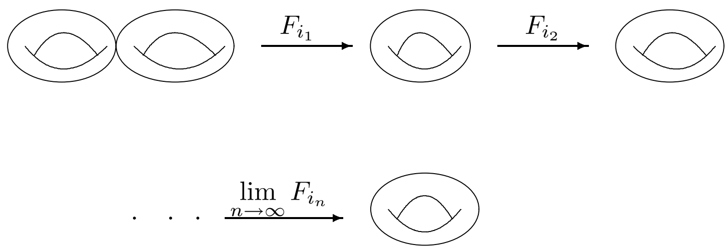

Fig.7

Corollary 2 *If $F_i : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1, i = 1, 2$ are two types of foldings, where $F_i(T_j^1) = T_i, j = 1, 2$. Then $\pi_1(\lim\limits_{n\to\infty} F_{i_n}(T_1^1 \vee T_2^1))$ is a free Abelian group of rank $2n$.*

Proof Since $F_i(T_j^1) = T_i, j = 1, 2$ we have the following chain $T_1^1 \vee T_2^1 \xrightarrow{F_{i_1}} T_i^n \xrightarrow{F_{i_2}} T_i^n \ldots \xrightarrow{\lim_{n\to\infty} F_{i_n}} T_i^n$ Since $\pi_1(T_i^n) = \underbrace{\pi_1(T_i \times T_i \times \ldots . \times T_i)}_{n-terms}$, , it follows that $\pi_1(T_i^n) \approx \underbrace{Z \times Z \times \ldots . \times Z}_{2n-terms}$. Hence ,$\pi_1(\lim\limits_{n\to\infty} F_{i_n}(T_1^1 \vee T_2^1))$ is a free Abelian of rank $2n$. □

Theorem 8 *If $F : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1$ is a folding by cut such that $F_1(T_1^1) \neq T_1, F_1(T_2^1) \neq T_1$ .Then there is induced folding $\overline{F}:\pi_1(T_1^1) * \pi_1(T_2^1) \longrightarrow \pi_1(T_1^1) * \pi_1(T_2^1)$ such that $\overline{F}(\pi_1(T_1^1) * \pi_1(T_2^1)) \approx Z * Z$.*

Proof Let $F : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1$ is a folding such that $F_1(T_1^1) \neq T_1, F_1(T_2^1) \neq T_1$as in Fig.8. Then, we can express each element $g = a_1 a_2 \cdots a_m, m \geq 1$ of $\pi_1(T_1^1) * \pi_1(T_2^1)$ in the

following forms

$$([\alpha_1]^{n_1},[\beta_1]^{k_1})([\alpha_2]^{n_2},[\beta_2]^{k_2})([\alpha_1]^{n_3},[\beta_1]^{k_3})\cdots([\alpha_1]^{n_{m-1}},[\beta_1]^{k_{m-1}})([\alpha_2]^{n_m}[\beta_2]^{k_m}),$$
$$([\alpha_1]^{n_1},[\beta_1]^{k_1})([\alpha_2]^{n_2},[\beta_2]^{k_2})([\alpha_1]^{n_3},[\beta_1]^{k_3})\cdots([\alpha_2]^{n_{m-1}},[\beta_2]^{k_{m-1}})([\alpha_1]^{n_m}[\beta_1]^{k_m}),$$
$$([\alpha_2]^{n_1},[\beta_2]^{k_1})([\alpha_1]^{n_2},[\beta_1]^{k_2})([\alpha_2]^{n_3},[\beta_2]^{k_3})\cdots([\alpha_2]^{n_{m-1}},[\beta_2]^{k_{m-1}})([\alpha_1]^{n_m}[\beta_1]^{k_m}),$$
$$([\alpha_2]^{n_1},[\beta_2]^{k_1})([\alpha_1]^{n_2},[\beta_1]^{k_2})([\alpha_2]^{n_3},[\beta_2]^{k_3})\cdots([\alpha_1]^{n_{m-1}},[\beta_1]^{k_{m-1}})([\alpha_1]^{n_m}[\beta_1]^{k_m}),$$

where $n_1,n_2,\cdots,n_m$, $k_1,k_2,\cdots,k_m$ are nonzero integers and

$$([\alpha_i]^{n_1},[\beta_i]^{k_1})\in\pi_1(T_1^1),([\alpha_i]^{n_2},[\beta_i]^{k_2})\in\pi_1(T_2^1).$$

Then, the induced folding of the element g is

$$\begin{aligned}
\overline{F}(g) &= \overline{F}([\alpha_1]^{n_1},[\beta_1]^{k_1})\overline{F}([\alpha_2]^{n_2},[\beta_2]^{k_2})\overline{F}([\alpha_1]^{n_3},[\beta_1]^{k_3})\\
&\quad\cdots\overline{F}([\alpha_1]^{n_{m-1}},[\beta_1]^{k_{m-1}})\overline{F}([\alpha_2]^{n_m}[\beta_2]^{k_m})\\
&= ([\alpha_1]^{n_1},[\beta_1]^{k_1})([\alpha_2]^{n_2},[\beta_2]^{k_2})([\alpha_1]^{n_3},[\beta_1]^{k_3})\cdots([\alpha_1]^{n_{m-1}},[\beta_1]^{k_{m-1}})([\alpha_2]^{n_m}[\beta_2]^{k_m}),\\
&\quad\overline{F}([\alpha_1]^{n_1},[\beta_1]^{k_1})\overline{F}([\alpha_2]^{n_2},[\beta_2]^{k_2})\overline{F}([\alpha_1]^{n_3},[\beta_1]^{k_3})\\
&\quad\cdots\overline{F}([\alpha_2]^{n_{m-1}},[\beta_2]^{k_{m-1}})\overline{F}([\alpha_1]^{n_m}[\beta_1]^{k_m})\\
&= ([\alpha_1]^{n_1},[\beta_1]^{k_1})([\alpha_2]^{n_2},[\beta_2]^{k_2})([\alpha_1]^{n_3},[\beta_1]^{k_3})\cdots([\alpha_2]^{n_{m-1}},[\beta_2]^{k_{m-1}})([\alpha_1]^{n_m}[\beta_1]^{k_m}),\\
&\quad\overline{F}([\alpha_2]^{n_1},[\beta_2]^{k_1})\overline{F}([\alpha_1]^{n_2},[\beta_1]^{k_2})\overline{F}([\alpha_2]^{n_3},[\beta_2]^{k_3})\\
&\quad\cdots\overline{F}([\alpha_2]^{n_{m-1}},[\beta_2]^{k_{m-1}})\overline{F}([\alpha_1]^{n_m}[\beta_1]^{k_m})\\
&= ([\alpha_2]^{n_1},[\beta_2]^{k_1})([\alpha_1]^{n_2},[\beta_1]^{k_2})([\alpha_2]^{n_3},[\beta_2]^{k_3})\cdots([\alpha_2]^{n_{m-1}},[\beta_2]^{k_{m-1}})([\alpha_1]^{n_m}[\beta_1]^{k_m}),\\
&\quad\overline{F}([\alpha_2]^{n_1},[\beta_2]^{k_1})\overline{F}([\alpha_1]^{n_2},[\beta_1]^{k_2})\overline{F}([\alpha_2]^{n_3},[\beta_2]^{k_3})\\
&\quad\cdots\overline{F}([\alpha_1]^{n_{m-1}},[\beta_1]^{k_{m-1}})\overline{F}([\alpha_1]^{n_m}[\beta_1]^{k_m})\\
&= ([\alpha_2]^{n_1},[\beta_2]^{k_1})([\alpha_1]^{n_2},[\beta_1]^{k_2})([\alpha_2]^{n_3},[\beta_2]^{k_3})....([\alpha_1]^{n_{m-1}},[\beta_1]^{k_{m-1}})([\alpha_1]^{n_m}[\beta_1]^{k_m}).
\end{aligned}$$

It follows from $[\widehat{\beta}_1],[\widehat{\beta}_2]\rightarrow 0$ (identity element) that there is an induced folding such that $\overline{F}:\pi_1(T_1^1)*\pi_1(T_2^1)\longrightarrow\pi_1(S_1^1)*\pi_1(S_2^1)$. Therefore, $\overline{F}(\pi_1(T_1^1)*\pi_1(T_2^1))\approx Z*Z$. □

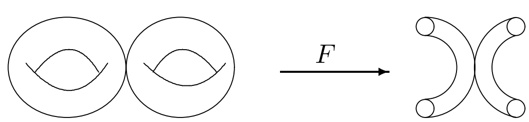

Fig.8

Corollary 3 *If $F_i:T_1^1\vee T_2^1\longrightarrow T_1^1\vee T_2^1, i=1,2$ are two types of foldings such that $F_i(T_j^1)=T_i^1, F_j(T_i^1)\neq T_i^1, i,j=1,2, i\neq j$. Then there are induced foldings $\overline{F_i}:\pi_1(T_1^1)*\pi_1(T_2^1)\longrightarrow\pi_1(T_1^1)*\pi_1(T_2^1)$. such that $\overline{F_i}(\pi_1(T_1^1)*\pi_1(T_2^1))\approx(Z\times Z)*Z$.*

Theorem 9 *If $F:T_1^1\vee T_2^1\longrightarrow T_1^1\vee T_2^1$ are a folding by cut such that $F(T_i^1)\neq T_i^1$, $for\ i=1,2$. Then $\pi_1(\lim_{n\to\infty}F_n(T_1^1\vee T_2^1))$, is a free group of rank ≤ 2 or identity group.*

Proof Consider ,$F(T_i^1)\neq T_i^1$, $for\ i=1,2$, then we have the following: $\lim_{n\to\infty}F_n(T_1^1\vee T_2^1)=S_1^1\vee S_2^1$ as in Fig.9(a) then ,$\pi_1(\lim_{n\to\infty}F_n(T_1^1\vee T_2^1))\approx\pi_1(S_1^1)\vee\pi_1(S_2^1)$, and so $\pi_1(\lim_{n\to\infty}F_n(T_1^1\vee T_2^1))$

$\approx Z * Z$. Hence, $\pi_1(\lim_{n\to\infty} F_n(T_1^1 \vee T_2^1))$ is a free group of rank 2.Also, If $\lim_{n\to\infty} F_n(T_1^1 \vee T_2^1)$ as in Fig.9(b), then $\pi_1(\lim_{n\to\infty} F_n(T_1^1 \vee T_2^1)) = 0$. Moreover, if $\lim_{n\to\infty} F_n(T_1^1 \vee T_2^1)$ as in Fig.9(c), then $\pi_1(\lim_{n\to\infty} F_n(T_1^1 \vee T_2^1)) \approx \pi_1(S_1^1) \approx Z$ .Therefore, $\pi_1(\lim_{n\to\infty} F_n(T_1^1 \vee T_2^1))$ is a free group of rank ≤ 2 or identity group. $\square$

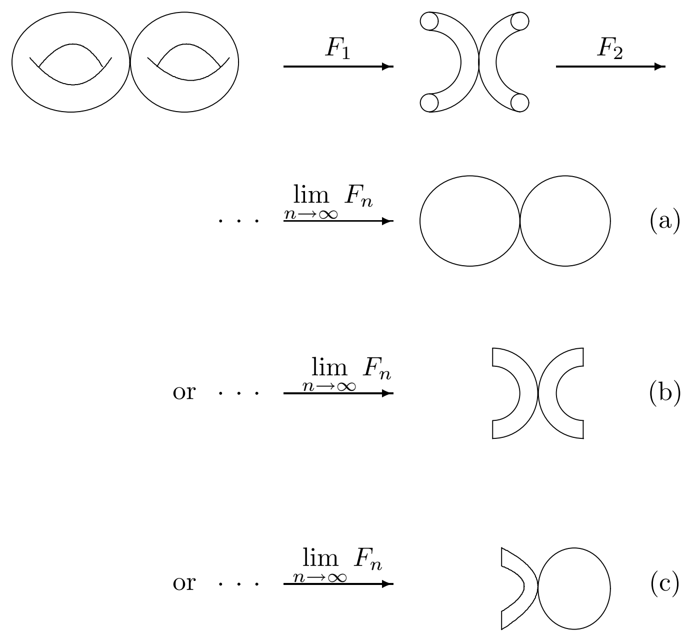

Fig.9

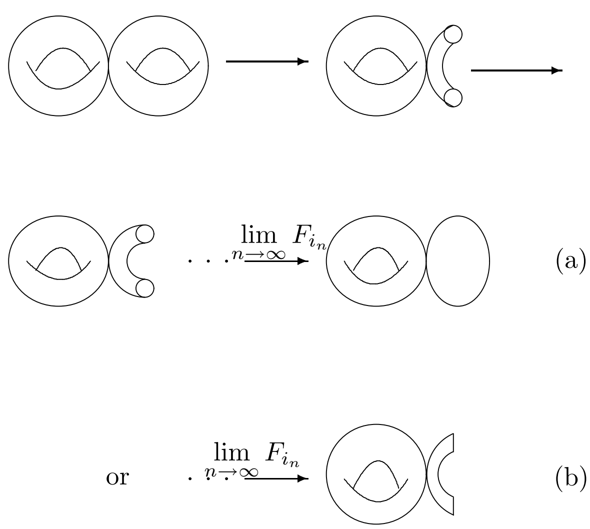

Fig.10

Theorem 10 *If* $F_i : T_1^1 \vee T_2^1 \longrightarrow T_1^1 \vee T_2^1$,$i = 1, 2$ *are two types of foldings such that*

$F_i(T_i^1) = T_i^1, F_j(T_i^1) \neq T_i^1, i, j = 1, 2, i \neq j$. Then $\pi_1(\lim\limits_{n\to\infty} F_n(T_1^1 \vee T_2^1))$ is either isomorphic $(Z \times Z) * Z$ to or $(Z \times Z)$.

Proof Since $F_i(T_i^1) = T_i^1, F_j(T_i^1) \neq T_i^1, i, j = 1, 2, i \neq j$, we have the following:

If $\lim\limits_{n\to\infty} F_{i_n}(T_1^1 \vee T_2^1) = T_i^1 \vee S_i^1$ as in Fig.10(a), then $\pi_1(\lim\limits_{n\to\infty} F_n(T_1^1 \vee T_2^1)) = \pi_1(T_i^1 \vee S_i^1) \approx (Z \times Z) * Z$. Also, if $\pi_1(\lim\limits_{n\to\infty} F_n(T_1^1 \vee T_2^1)) = \pi_1(T_i^1)$ as in Fig.10(b) then $\pi_1(\lim\limits_{n\to\infty} F_n(T_1^1 \vee T_2^1)\pi_1(T_i^1) \approx Z \times Z$. Hence, $\pi_1(\lim\limits_{n\to\infty} F_n(T_1^1 \vee T_2^1))$ is either isomorphic to $(Z \times Z) * Z$ or

$(Z \times Z)$. □

Theorem 11 *If $F : T_1^n \vee T_2^n \longrightarrow T_1^n \vee T_2^n$ is a folding such that $F(T_1^n) = T_1^n$ and $F(T_2^n) \neq T_2^n$ where $F(T_2^n) = \underbrace{F(T_2^1) \times F(T_2^1) \times .... \times F(T_2^1)}_{n-terms}$,$F(T_2^1) \neq T_2^1$ is a folding by cut. Then, $\pi_1(\lim\limits_{n\to\infty} F_n(T_1^n \vee T_2^n))$ is isomorphic to $\underbrace{(Z \times Z \times .... \times Z)}_{2n-terms} * \underbrace{Z \times Z \times .... \times Z}_{n-terms}$.*

Proof Since $F(T_1^n) = T_1^n$, $F(T_2^n) \neq T_2^n$ we have the following chain:

$$T_1^n \vee T_2^n \xrightarrow{F} T_1^n \vee \underbrace{F(S_1^1) \times S_2^1 \times F(S_1^1) \times S_2^1 \times \cdots \times F(S_1^1) \times S_2^1}_{2n-terms} \xrightarrow{F},$$

$$T_1^n \vee T_2^n \xrightarrow{F} T_1^n \vee \underbrace{F(S_1^1) \times S_2^1 \times F(S_1^1) \times S_2^1 \times \cdots \times F(S_1^1) \times S_2^1}_{2n-terms} \xrightarrow{F},$$

$$T_1^n \vee \underbrace{F(F(S_1^1)) \times S_2^1 \times F(F(S_1^1)) \times S_2^1 \times \cdots \times F(F(S_1^1)) \times S_2^1}_{2n-terms} \xrightarrow[n\to\infty]{\lim F_n},$$

$$T_1^n \vee \underbrace{(S_2^1 \times S_2^1 \times \cdots \times S_2^1)}_{n-terms}.$$

Hence, $\pi_1(\lim\limits_{n\to\infty} F_n(T_1^n \vee T_2^n))$ is isomorphic to $\underbrace{(Z \times Z \times \cdots \times Z)}_{2n-terms} * \underbrace{Z \times Z \times \cdots \times Z}_{n-terms}$. □

Theorem 12 *Let $F : M \to M$ is a folding by cut or with singularity , and M is a manifold homeomorphic to S^1 or T^1. Then, there are unfoldings $unf : F(M) \subset M \to M$ such that $\pi_1(\lim\limits_{n\to\infty} unf_n(F(M))$ is isomorphic to Z or $Z \times Z$.*

Proof We have two cases following.

Case 1. Let M be a manifold homeomorphic to S^1 , if $F : S^1 \to S^1$ is a folding by cut.

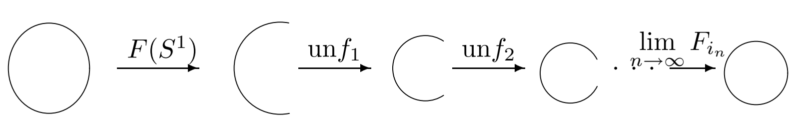

Fig.11

Then, we can define a sequence of unfoldings

$unf_1 : F(S^1) \to M_1, F(S^1) \neq S^1, M_1 \subseteq S^1,\ \ unf_2 : M_1 \to M_2, ..., unf_n : M_1 \to M_2,$

$\lim\limits_{n\to\infty} unf_n(F(M)) = S^1$ as in Fig.11. Thus $\pi_1(\lim\limits_{n\to\infty} unf_n(F(M)) \approx Z$.

Case 2. Let M be a manifold homeomorphic to T^1 , if $F : T^1 \to T^1$ is a folding such that $F(S_1^1) = S_1^1$ and $F(S_2^1) \neq S_2^1$. So we can define a sequence of unfoldings following.

$unf_1 : F(T^1) \to M_1,\ unf_2 : M_1 \to M_2, \cdots,\ unf_n : M_1 \to M_2,$

$\lim\limits_{n\to\infty} unf_n(F(M)) = T^1$ as in Fig.12. Thus $\pi_1(\lim\limits_{n\to\infty} unf_n(F(M)) \approx Z \times Z$.

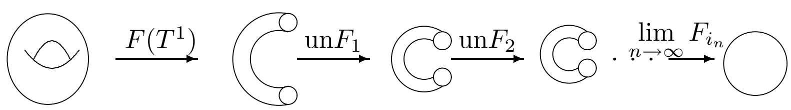

Fig.12

Therefore, $\pi_1(\lim\limits_{n\to\infty} unf_n(F(M))$ is isomorphic to Z or $Z \times Z$. □

Corollary 4 *Let $F : M \to M$ be a folding by cut or with singularity, M is a manifold homeomorphic to S^n or $T^n, n \geq 2$. Then there are unfoldings $unf : F(M) \subset M \to M$ such that $\pi_1(\lim\limits_{n\to\infty} unf_n(F(M))$ is the identity group or a free Abelian group of rank $2n$.*

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Absolutely Harmonious Labeling of Graphs

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Abstract: Absolutely harmonious labeling f is an injection from the vertex set of a graph G with q edges to the set $\{0, 1, 2, ..., q-1\}$, if each edge uv is assigned $f(u)+f(v)$ then the resulting edge labels can be arranged as $a_0, a_1, a_2, ..., a_{q-1}$ where $a_i = q-i \ or \ q+i, \ 0 \leq i \leq q-1$. However, when G is a tree one of the vertex labels may be assigned to exactly two vertices. A graph which admits absolutely harmonious labeling is called *absolutely harmonious graph*. In this paper, we obtain necessary conditions for a graph to be absolutely harmonious and study absolutely harmonious behavior of certain classes of graphs.

Key Words: Graph labeling, Smarandachely k-labeling, harmonious labeling, absolutely harmonious labeling.

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§1. Introduction

A vertex labeling of a graph G is an assignment f of labels to the vertices of G that induces a label for each edge xy depending on the vertex labels. For an integer $k \geq 1$, a *Smarandachely k-labeling* of a graph G is a bijective mapping $f : V \to \{1, 2, \cdots, k|V(G)| + |E(G)|\}$ with an additional condition that $|f(u) - f(v)| \geq k$ for $\forall uv \in E$. particularly, if $k = 1$, i.e., such a Smarandachely 1-labeling is the usually labeling of graph. Among them, labelings such as those of *graceful labeling, harmonious labeling* and *mean labeling* are some of the interesting vertex labelings found in the dynamic survey of graph labeling by Gallian [2]. Harmonious labeling is one of the fundamental labelings introduced by Graham and Sloane [3] in 1980 in connection with their study on error correcting code. *Harmonious labeling* f is an injection from the vertex set of a graph G with q edges to the set $\{0, 1, 2, ..., q-1\}$, if each edge uv is assigned $f(u)+f(v)(mod \ q)$ then the resulting edge labels are distinct. However, when G is a tree one of the vertex labels may be assigned to exactly two vertices. Subsequently a few variations of harmonious labeling, namely, *strongly c-harmonious labeling* [1], *sequential labeling* [5], *elegant labeling* [1] and *felicitous labeling* [4] were introduced. The later three labelings were introduced to avoid such exceptions for the trees given in harmonious labeling. A strongly

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1-harmonious graph is also known as strongly harmonious graph.

It is interesting to note that if a graph G with q edges is harmonious then the resulting edge labels can be arranged as $a_0, a_1, a_2, \cdots, a_{q-1}$ where $a_i = i$ *or* $q+i$, $0 \leq i \leq q-1$. That is for each i, $0 \leq i \leq q-1$ there is a distinct edge with label either i or $q+i$. An another interesting and natural variation of edge label could be $q-i$ or $q+i$. This prompts to define a new variation of harmonious labeling called *absolutely harmonious labeling.*

Definition 1.1 *An absolutely harmonious labeling f is an injection from the vertex set of a graph G with q edges to the set $\{0, 1, 2, ..., q-1\}$, if each edge uv is assigned $f(u)+f(v)$ then the resulting edge labels can be arranged as $a_0, a_1, a_2, ..., a_{q-1}$ where $a_i = q-i$ or $q+i$, $0 \leq i \leq q-1$. However, when G is a tree one of the vertex labels may be assigned to exactly two vertices. A graph which admits absolutely harmonious labeling is called absolutely harmonious graph.*

The result of Graham and Sloane [3] states that $C_n, n \cong 1(mod\ 4)$ is harmonious, but we show that $C_n, n \cong 1(mod\ 4)$ is not an absolutely harmonious graph. On the other hand, we show that C_4 is an absolutely harmonious graph, but it is not harmonious. We observe that a strongly harmonious graph is an absolutely harmonious graph.

To initiate the investigation on absolutely harmonious graphs, we obtain necessary conditions for a graph to be an absolutely harmonious graph and prove the following results:

1. Path $P_n, n \geq 3$, a class of banana trees, and $P_n \odot K_m^c$ are absolutely harmonious graphs.
2. Ladders, $C_n \odot K_m^c$, Triangular snakes, Quadrilateral snakes, and mK_4 are absolutely harmonious graphs.
3. Complete graph K_n is absolutely harmonious if and only if $n = 3$ or 4.
4. Cycle $C_n, n \cong 1$ *or* 2 $(mod\ 4)$, $C_m \times C_n$ where m and n are odd, $mK_3, m \geq 2$ are not absolutely harmonious graphs.

§2. Necessary Conditions

Theorem 2.1 *If G is an absolutely harmonious graph, then there exists a partition (V_1, V_2) of the vertex set $V(G)$, such that the number of edges connecting the vertices of V_1 to the vertices of V_2 is exactly $\left\lceil \frac{q}{2} \right\rceil$.*

Proof If G is an absolutely harmonious graph,then the vertices can be partitioned into two sets V_1 and V_2 having respectively even and odd vertex labels. Observe that among the q edges $\frac{q}{2}$ edges or $\left\lceil \frac{q}{2} \right\rceil$ edges are labeled with odd numbers, according as q is even or q is odd. For an edge to have odd label, one end vertex must be odd labeled and the other end vertex must be even labeled. Thus, the number of edges connecting the vertices of V_1 to the vertices of V_2 is exactly $\left\lceil \frac{q}{2} \right\rceil$. □

Remark 2.2 A simple and straight forward application of Theorem 2.1 identifies the non absolutely harmonious graphs. For example, complete graph K_n has $\frac{n(n-1)}{2}$ edges. If we assign

m vertices to the part V_1, there will be $m(n-m)$ edges connecting the vertices of V_1 to the vertices of V_2. If K_n has an absolutely harmonious labeling, then there is a choice of m for which $m(n-m) = \left\lceil \frac{n^2-n}{4} \right\rceil$. Such a choice of m does not exist for $n = 5, 7, 8.10, ...$

A graph is called *even graph* if degree of each vertex is even.

Theorem 2.3 *If an even graph G is absolutely harmonious then $q \cong 0$ or 3 (mod 4).*

Proof Let G be an even graph with $q \cong 1$ *or* 2 (mod 4) and $d(v)$ denotes the degree of the vertex v in G. Suppose f be an absolutely harmonious labeling of G. Then the resulting edge labels can be arranged as $a_0, a_1, a_2, ..., a_{q-1}$ where $a_i = q - i$ *or* $q + i$, $0 \leq i \leq q-1$. In other words, for each i, the edge label a_i is $(q-i) + 2ib_i$, $0 \leq i \leq q-1$ where $b_i \in \{0, 1\}$. Evidently

$$\sum_{v \in V(G)} d(v)f(v) - 2\sum_{k=0}^{q-1} kb_k = \binom{q+1}{2}.$$

As $d(v)$ is even for each v and $q \cong 1$ *or* 2 (mod 4),

$$\sum_{v \in V(G)} d(v)f(v) - 2\sum_{k=0}^{q-1} kb_k \cong 0 \ (mod\ 2)$$

but $\binom{q+1}{2} \cong 1$ (mod 2). This contradiction proves the theorem. □

Corollary 2.4 *A cycle C_n is not an absolutely harmonious graph if $n \cong 1$ or 2 (mod 4).*

Corollary 2.5 *A grid $C_m \times C_n$ is not an absolutely harmonious graph if m and n are odd.*

Theorem 2.6 *If f is an absolutely harmonious labeling of the cycle C_n , then edges of C_n can be partitioned into two sub sets E_1, E_2 such that*

$$\sum_{uv \in E_1} |f(u) + f(v) - n| = \frac{n(n+1)}{4} \quad and \quad \sum_{uv \in E_2} |f(u) + f(v) - n| = \frac{n(n-3)}{4}.$$

Proof Let $v_1v_2v_3...v_nv_1$ be the cycle C_n, where $e_i = v_{i-1}v_i, 2 \leq i \leq n$ and $e_1 = v_nv_1$. Define $E_1 = \{uv \in E/ f(u) + f(v) - n \text{ is non negative}\}$ and $E_2 = \{uv \in E/ f(u) + f(v) - n \text{ is negative}\}$. Since f is an absolutely harmonious labeling of the cycle C_n,

$$\sum_{uv \in E} |f(u) + f(v) - n| = \frac{n(n-1)}{2}.$$

In other words,

$$\sum_{uv \in E_1} |f(u) + f(v) - n| + \sum_{uv \in E_2} |f(u) + f(v) - n| = \frac{n(n-1)}{2}. \tag{1}$$

Since $\sum_{uv \in E}(f(u) + f(v) - n) = -n$, we have

$$\sum_{uv \in E_1} |f(u) + f(v) - n| - \sum_{uv \in E_2} |f(u) + f(v) - n| = -n. \tag{2}$$

Solving equations (1) and (2), we get the desired result. □

Remark 2.7 If $n \cong 1$ *or* 2 $(mod\ 4)$ then both $\dfrac{n(n+1)}{4}$ and $\dfrac{n(n-3)}{4}$ cannot be integers. Thus the cycle C_n is not an absolutely harmonious graph if $n \cong 1$ *or* 2 $(mod\ 4)$.

Remark 2.8 Observe that the conditions stated in Theorem 2.1, Theorem 2.3, and Theorem 2.6 are necessary but not sufficient. Note that C_8 satisfies all the conditions stated in Theorems 2.1, 2.3, and 2.6 but it is not an absolutely harmonious graph. For, checking each of the $\dfrac{8!}{2}$ possibilities reveals the desired result about C_8.

§3. Absolutely Harmonious Graphs

Theorem 3.1 *The path P_{n+1}, where $n \geq 2$ is an absolutely harmonious graph.*

Proof Let $P_{n+1} : v_1v_2...v_{n+1}$ be a path, $r = \left\lceil \dfrac{n}{2} \right\rceil$, $s = \begin{cases} \lceil \frac{r}{2} \rceil + 1 & \text{if } n \cong 0\ (mod\ 4) \\ \lceil \frac{r}{2} \rceil & \text{otherwise} \end{cases}$,
$t = \begin{cases} s-1 & \text{if } n \cong 0 \text{ or } 1\ (mod\ 4) \\ s & \text{if } n \cong 2 \text{ or } 3\ (mod\ 4) \end{cases}$, $T_1 = n$, $T_2 = \begin{cases} 2t+2 & \text{if } n \cong 0 \text{ or } 1\ (mod\ 4) \\ 2t+1 & \text{if } n \cong 2 \text{ or } 3\ (mod\ 4) \end{cases}$ and $T_3 = \begin{cases} -1 & \text{if } n \cong 0 \text{ or } 1\ (mod\ 4) \\ -2 & \text{if } n \cong 2 \text{ or } 3\ (mod\ 4) \end{cases}$.

Then $r + s + t = n + 1$. Define $f : V(P_{n+1}) \to \{0, 1, 2, 3, \cdots, n-1\}$ by:

$f(v_i) = T_1 - i$ *if* $1 \leq i \leq r$, $f(v_{r+i}) = T_2 - 2i$ *if* $1 \leq i \leq s$ *and* $f(v_{r+s+i}) = T_3 + 2i$ *if* $1 \leq i \leq t$.

Evidently f is an absolutely harmonious labeling of P_{n+1}. For example, an absolutely harmonious labeling of P_{12} is shown in Fig.3.1. □

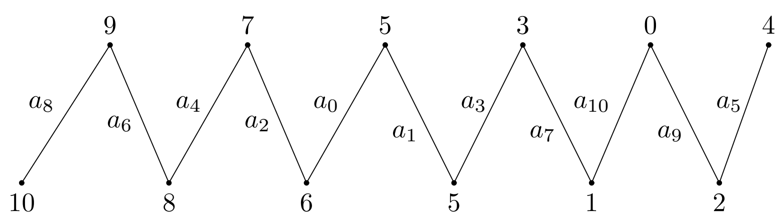

Fig.3.1

The tree obtained by joining a new vertex v to one pendant vertex of each of the k disjoint stars $K_{1,n_1}, K_{1,n_2}, K_{1,n_3}, ..., K_{1,n_k}$ is called a *banana tree*. The class of all such trees is denoted by $BT(n_1, n_2, n_3, ..., n_k)$.

Theorem 3.2 *The banana tree $BT(n, n, n, ..., n)$ is absolutely harmonious.*

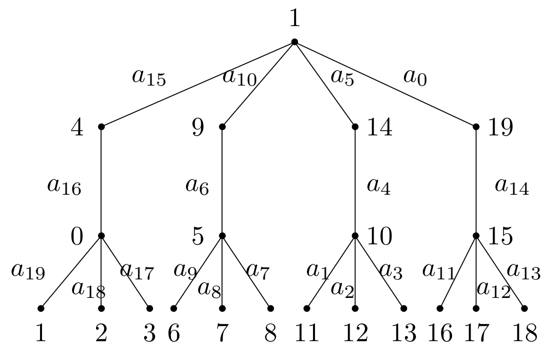

Fig.3.2

Proof Let $V(BT(n,n,n,\cdots,n) = \{v\} \cup \{v_j, v_{jr} : 1 \le j \le k \ and \ 1 \le r \le n\}$ where $d(v_j) = n$ and $E(BT(n,n,n,...,n) = \{vv_{jn} : 1 \le j \le k\} \cup \{v_j v_{jr} : 1 \le j \le k, 1 \le r \le n\}$. Clearly $BT(n,n,\cdots,n)$ has order $(n+1)k+1$ and size $(n+1)k$. Define

$$f : V(BT(n,n,\cdots,n) \to \{1,2,3,...,(n+1)k-1\}$$

as follows:

$$f(v) = 1,\ f(v_j) = (n+1)(j-1) : 1 \le j \le k,\ f(v_{jr}) = (n+1)(j-1) + r : 1 \le r \le n.$$

It can be easily verified that f is an absolutely harmonious labeling of $BT(n,n,n,...,n)$. For example an absolutely harmonious labeling of $BT(4,4,4,4)$is shown in Fig.3.2. □

The *corona* $G_1 \odot G_2$ of two graphs $G_1(p_1,q_1)$ and $G_2(p_2,q_2)$ is defined as the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 to all the vertices in the i^{th} copy of G_2.

Theorem 3.3 *The corona $P_n \odot K_m^C$ is absolutely harmonious.*

Proof Let $V(P_n \odot K_m^C) = \{u_i : 1 \le i \le n\} \cup \{u_{ij} : 1 \le i \le n, 1 \le j \le m\}$ and $E(P_n \odot K_m^C) = \{u_i u_{i+1} : 1 \le i \le n-1\} \cup \{u_i u_{ij} : 1 \le i \le n, 1 \le j \le m\}$. We observe that $P_n \odot K_m^C$ has order $(m+1)n$ and size $(m+1)n-1$. Define $f : V(P_n \odot K_m^C) \longrightarrow \{0,1,2,...,mn+n-2\}$ as follows:

$$f(u_i) = \begin{cases} 0 & if\ i = 1, \\ (m+1)(i-1) & \text{if } i = \lceil \frac{n}{2} \rceil \\ (m+1)(i-1)-1 & \text{otherwise,} \end{cases} \qquad f(u_{im}) = \begin{cases} (m+1)i & \text{if } 1 \le i \le \lceil \frac{n}{2} \rceil - 2, \\ (m+1)i - 1 & \text{if } i = \lceil \frac{n}{2} \rceil - 1, \\ (m+1)i - 2 & \lceil \frac{n}{2} \rceil \le i \le n, \end{cases}$$

and for $1 \le j \le m-1$,

$$f(u_{ij}) = \begin{cases} (m+1)(i-1)+j & \text{if } 1 \le i \le \lceil \frac{n}{2} \rceil - 1, \\ (m+1)(i-1)+j-1 & \text{if } \lceil \frac{n}{2} \rceil \le i \le n. \end{cases}$$

It can be easily verified that f is an absolutely harmonious labeling of $P_n \odot K_m^C$. For example an absolutely harmonious labeling of $P_5 \odot K_3^C$ is shown in Fig. 3.3. □

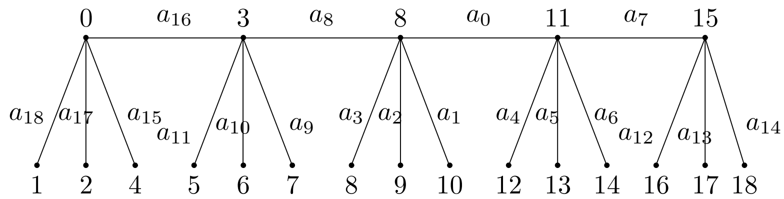

Fig.3.3

Theorem 3.4 *The corona $C_n \odot K_m^C$ is absolutely harmonious.*

Proof Let $V(C_n \odot K_m^C) = \{u_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(C_n \odot K_m^C) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{u_i u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$. We observe that $C_n \odot K_m^C$ has order $(m+1)n$ and size $(m+1)n$. Define $f : V(C_n \odot K_m^C) \longrightarrow \{0, 1, 2, ..., mn + n - 1\}$ as follows:

$$f(u_i) = \begin{cases} 0 & \text{if } i = 1, \\ (m+1)(i-1) - 1 & \text{if } 2 \leq i \leq \frac{n-1}{2}, \\ (m+1)(i-1) & \text{otherwise}, \end{cases}, \quad f(u_{im}) = \begin{cases} (m+1)i & \text{if } 1 \leq i \leq \frac{n-3}{2}, \\ (m+1)i - 1 & \text{otherwise} \end{cases}$$

and for $1 \leq j \leq m-1$

$$f(u_{ij}) = \begin{cases} (m+1)(i-1) + j & \text{if } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1, \\ (m+1)(i-1) + j - 1 & \text{if } \left\lceil \frac{n}{2} \right\rceil \leq i \leq n. \end{cases}$$

It can be easily verified that f is an absolutely harmonious labeling of $C_n \odot K_m^C$. For example an absolutely harmonious labeling of $C_5 \odot K_3^C$ is shown in Figure 3.4. $\square$

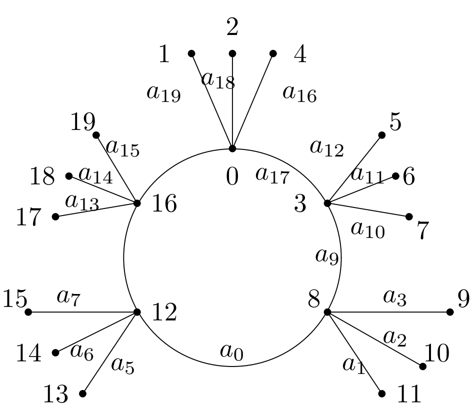

Fig.3.4

Theorem 3.5 *The ladder $P_n \times P_2$, where $n \geq 2$ is an absolutely harmonious graph.*

Proof Let $V(P_n \times P_2) = \{u_1, u_2, u_3, ..., u_n\} \cup \{v_1, v_2, v_3, ..., v_n\}$ and $E(P_n \times P_2) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}$. We note that $P_n \times P_2$ has order

$2n$ and size $3n-2$.

Case 1. $n \equiv 0(mod\ 4)$.

Define $f : V(P_n \times P_2) \longrightarrow \{0, 1, 2, ..., 3n-3\}$ by

$$f(u_i) = \begin{cases} 3i-2 & \text{if } i \text{ is odd,} \\ 3i-2 & \text{if } i \text{ is even and } 2 \le i \le \frac{n-4}{2}, \\ 3i-1 & \text{if } i \text{ is even and } i = \frac{n}{2}, \\ 3i-3 & \text{if } i \text{ is even and } \frac{n+4}{2} \le i \le n, \end{cases}$$

$$f(v_1) = 0, \quad f(v_{\frac{n+2}{2}}) = \frac{3n-6}{2}, \quad f(v_{i+1}) = f(u_i) + 1 \text{ if } 1 \le i \le n-1 \text{ and } i \ne \frac{n}{2}.$$

Case 2. $n \equiv 1(mod\ 4)$.

Define $f : V(P_n \times P_2) \longrightarrow \{0, 1, 2, ..., 3n-3\}$ by

$$f(u_i) = \begin{cases} 3i-2 & \text{if } i \text{ is odd and } 1 \le i \le \frac{n-3}{2}, \\ 3i-1 & \text{if } i = \frac{n+1}{2}, \\ 3i-3 & \text{if } i \text{ is odd and } \frac{n+5}{2} \le i \le n, \\ 3i-2 & \text{if } i \text{ is even,} \end{cases}$$

$$f(v_1) = 0, \quad f(v_{\frac{n+3}{2}}) = \tfrac{3n-3}{2}, \quad f(v_{i+1}) = f(u_i) + 1 \text{ if } 1 \le i \le n-1 \text{ and } i \ne \tfrac{n+1}{2}.$$

Case 3. $n \equiv 2(mod\ 4)$.

Define $f : V(P_n \times P_2) \longrightarrow \{0, 1, 2, ..., 3n-3\}$ by

$$f(u_i) = \begin{cases} 3i-2 & \text{if } i \text{ is odd,} \\ 3i-2 & \text{if } i \text{ is even and } 2 \le i \le \frac{n-2}{2}, \\ 3i-3 & \text{if } i \text{ is even and } \frac{n+2}{2} \le i \le n, \end{cases}$$

$$f(v_1) = 0, \quad f(v_{i+1}) = f(u_i) + 1 \text{ if } 1 \le i \le n-1.$$

Case 4. $n \equiv 3(mod\ 4)$.

Define $f : V(P_n \times P_2) \longrightarrow \{0, 1, 2, ..., 3n-3\}$ by

$$f(u_i) = \begin{cases} 3i-2 & \text{if } i \text{ is odd and } 1 \le i \le \frac{n-1}{2}, \\ 3i-3 & \text{if } i \text{ is odd and } \frac{n+3}{2} \le i \le n, \\ 3i-2 & \text{if } i \text{ is even.} \end{cases}$$

$$f(v_1) = 0, \quad f(v_{i+1}) = f(u_i) + 1 \text{ if } 1 \le i \le n-1.$$

In all four cases, it can be easily verified that f is an absolutely harmonious labeling of $P_n \times P_2$. For example, an absolutely harmonious labeling of $P_9 \times P_2$ is shown in Fig.3.5. □

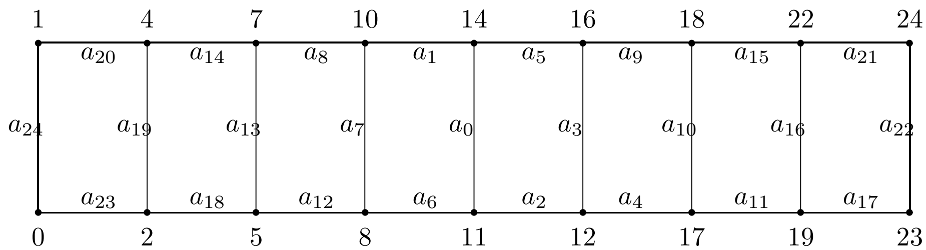

Fig.3.5

A K_n-*snake* has been defined as a connected graph in which all blocks are isomorphic to K_n and the block-cut point graph is a path. A K_3-snake is called *triangular snake*.

Theorem 3.6 *A triangular snake with n blocks is absolutely harmonious if and only if $n \cong 0$ or 1 $(mod\ 4)$.*

Proof The necessity follows from Theorem 2.3.Let G_n be a triangular snake with n blocks on p vertices and q edges. Then $p = 2n-1$ and $q = 3n$. Let $V(G_n) = \{u_i : 1 \le i \le n+1\} \cup \{v_i : 1 \le i \le n\}$ and $E(G_n) = \{u_iu_{i+1}, u_iv_i, u_{i+1}v_i : 1 \le i \le n\}$.

Case 1. $n \equiv 0\ (mod\ 4)$.

Let $m = \dfrac{n}{4}$. Define $f : V(G_n) \longrightarrow \{0, 1, 2, ..., 3n-1\}$ as follows:

$$f(u_i) = \begin{cases} 0 & \text{if } i = 1, \\ 2i-2 & \text{if } 2 \le i \le 3m \text{ and } i \equiv 0 \text{ or } 2\ (mod\ 3), \\ 2i-1 & \text{if } 2 \le i \le 3m \text{ and } i \equiv 1\ (mod\ 3), \\ 6i-3n-7 & \text{otherwise,} \end{cases}$$

$$f(v_i) = \begin{cases} 1 & \text{if } i = 1, \\ 2i-1 & \text{if } 2 \le i \le 3m-1 \text{ and } i \equiv \ \text{ or } 2\ (mod\ 3), \\ 2i-2 & \text{if } 2 \le i \le 3m-1 \text{ and } i \equiv 1\ (mod\ 3), \\ 6m+1 & \text{if } i = 3m, \\ 6i-3n-3 & \text{otherwise.} \end{cases}$$

Case 2. $n \equiv 1\ (mod\ 4)$.

Let $m = \dfrac{n-1}{4}$. Define $f : V(G_n) \longrightarrow \{0, 1, 2, ..., 3n-1\}$ as follows:

$$f(u_i) = \begin{cases} 0 & \text{if } i = 1, \\ 2i-2 & \text{if } 2 \leq i \leq 3m+2 \text{ and } i \equiv 0 \ or\ 2\ (mod\ 3), \\ 2i-1 & \text{if } 2 \leq i \leq 3m+2 \text{ and } i \equiv 1\ (mod\ 3), \\ 6i-3n-7 & \text{otherwise,} \end{cases}$$

$$f(v_i) = \begin{cases} 1 & \text{if } i = 1, \\ 2i-1 & \text{if } 2 \leq i \leq 3m+1 \text{ and } i \equiv 0 \ or\ 2\ (mod\ 3) \\ 2i-2 & \text{if } 2 \leq i \leq 3m+1 \text{ and } i \equiv 1\ (mod\ 3) \\ 6i-3n-3 & \text{otherwise.} \end{cases}$$

In both cases, it can be easily verified that f is an absolutely harmonious labeling of the triangular snake G_n. For example, an absolutely harmonious labeling of a triangular snake with five blocks is shown in Fig.3.6. □

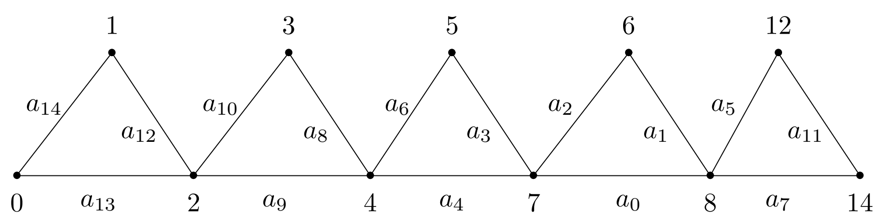

Fig.3.6

Theorem 3.7 *K_4-snakes are absolutely harmonious.*

Proof Let G_n be a K_4-snake with n blocks on p vertices and q edges. Then $p = 3n+1$ and $q = 6n$. Let $V(G_n) = \{u_i, v_i, w_i : 1 \leq i \leq n\} \cup \{v_{n+1}\}$ and $E(G_n) = \{u_iv_i, u_iw_i, v_iw_i : 1 \leq i \leq n\} \cup \{u_iv_{i+1}, v_iv_{i+1}, w_iv_{i+1} : 1 \leq i \leq n\}$ Define $f : V(G_n) \longrightarrow \{0, 1, 2, ..., 6n-1\}$ as follows:

$$f(u_i) = 3i-3, f(v_i) = 3i-2, f(w_i) = 3i-1$$

where $1 \leq i \leq n$, and $f(v_{n+1}) = 3n+1$. It can be easily verified that f is an absolutely harmonious labeling of G_n and hence K_4-snakes are absolutely harmonious. For example, an absolutely harmonious labeling of a K_4-snake with five blocks is shown in Fig.3.7. □

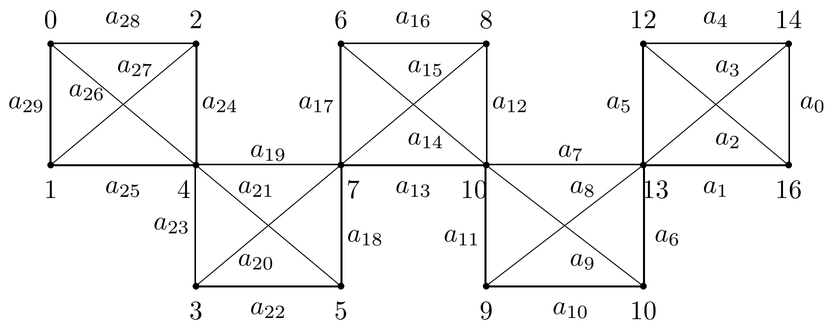

Fig.3.7

A *quadrilateral snake* is obtained from a path $u_1u_2...u_{n+1}$ by joining u_i, u_{i+1} to new vertices v_i, w_i respectively and joining v_i and w_i.

Theorem 3.8 *All quadrilateral snakes are absolutely harmonious.*

Proof Let G_n be a quadrilateral snake with $V(G_n) = \{u_i : 1 \leq i \leq n+1\} \cup \{v_i, w_i : 1 \leq i \leq n\}$ and $E(G_n) = \{u_iu_{i+1}, u_iv_i, u_{i+1}w_i, v_iw_i : 1 \leq i \leq n\}$. Then $p = 3n+1$ and $q = 4n$. Let
$$m = \begin{cases} \frac{n}{2} & if\ n \equiv 0\ (mod\ 2) \\ \frac{n-1}{2} & if\ n \equiv 1\ (mod\ 2) \end{cases}.$$

Define $f : V(G_n) \longrightarrow \{0, 1, 2, ...4n-1\}$ as follows:

$$f(u_i) = \begin{cases} 0 & \text{if } i = 1, \\ 4i-6 & \text{if } 2 \leq i \leq m+1, \\ 4i-7 & \text{if } m+2 \leq i \leq n+1 \end{cases}, \qquad f(v_i) = \begin{cases} 4i-3 & \text{if } 1 \leq i \leq m, \\ 4i-2 & \text{if } m+1 \leq i \leq n, \end{cases}$$

$$f(w_i) = \begin{cases} 4i & \text{if } 1 \leq i \leq m, \\ 4i-1 & \text{if } m+1 \leq i \leq n. \end{cases}$$

It can be easily verified that f is an absolutely harmonious labeling of the quadrilateral snake G_n and hence quadrilateral snakes are absolutely harmonious. For example, an absolutely harmonious labeling of a quadrilateral snake with six blocks is shown in Fig.3.8. □

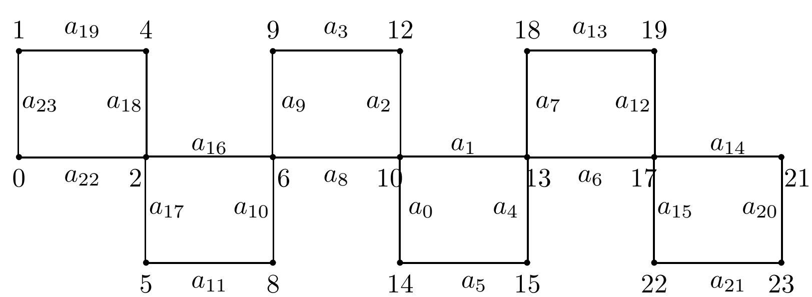

Fig.3.8

Theorem 3.9 *The disjoint union of m copies of the complete graph on four vertices, mK_4 is absolutely harmonious.*

Proof Let u_i^j where $1 \le i \le 4$ and $1 \le j \le m$ denotes the i^{th} vertex of the j^{th} copy of mK_4. We note that that mK_4 has order $4m$ and size $6m$. Define $f : V(mK_4) \longrightarrow \{0, 1, 2, ...6m-1\}$ as follows:$f(u_1^1) = 0, f(u_2^1) = 1, f(u_3^1) = 2, f(u_4^1) = 4, f(u_1^2) = q-3, f(u_2^2) = q-4, f(u_3^2) = q-5, f(u_4^2) = q-7, f(u_i^{j+2}) = f(u_i^j) + 6\ if\ j\ is\ odd, and\ f(u_i^{j+2}) = f(u_i^j) - 6\ if\ j\ is\ even,$ where $1 \le i \le 4$ and $1 \le j \le m-2$. Clearly f is an absolutely harmonious labeling. For example, an absolutely harmonious labeling of $5K_4$ is shown in Figure 11. *Box*

Observation 3.10 If f is an absolutely harmonious labeling of a graph G,which is not a tree, then

1. Each x in the set $\{0, 1, 2\}$ has inverse image.
2. Inverse images of 0 and 1 are adjacent in G.
3. Inverse images of 0 and 2 are adjacent in G.

Theorem 3.11 *The disjoint union of m copies of the complete graph on three vertices, mK_3 is absolutely harmonious if and only if $m = 1$.*

Proof Let u_i^j,where$1 \le i \le 3$ and $1 \le j \le m$ denote the i^{th} vertex of the j^{th} copy of mK_3. Assignments of the values $0, 1, 2$ to the vertices of K_3 gives the desired absolutely harmonious labeling of K_3. For $m \ge 2$, mK_3 has $3m$ vertices and $3m$ edges. If mK_3 is an absolutely harmonious graph, we can assign the numbers $\{0, 1, 2,\ 3m-1\}$ to the vertices of mK_3 in such a way that its edges receive each of the numbers a_0,a_1,...,a_{q-1} where $a_i = q-i\ or\ q+i$, $0 \le i \le q-1$. By Observation 3.10, we can assume, without loss of generality that $f(u_1^1) = 0, f(u_2^1) = 1, f(u_3^1) = 2$. Thus we get the edge labels a_{q-1}, a_{q-2} and a_{q-3}. In order to have an edge labeled a_{q-4}, we must have two adjacent vertices labeled $q-1$ and $q-3$. we can assume without loss of generality that $f(u_1^2) = q-1$ and $f(u_2^2) = q-3$. In order to have an edge labeled a_{q-5}, we must have $f(u_2^3) = q-4$. There is now no way to obtain an edge labeled a_{q-6}. This contradiction proves the theorem. □

Theorem 3.12 *A complete graph K_n is absolutely harmonious graph if and only if $n = 3\ or\ 4$.*

Proof From the definition of absolutely harmonious labeling, it can be easily verified that K_1 and K_2 are not absolutely harmonious graphs. Assignments of the values $0, 1, 2$ and $0, 1, 2, 4$ respectively to the vertices of K_3 and K_4 give the desired absolutely harmonious labeling of them. For $n > 4$, the graph K_n has $q \ge 10$ edges. If K_n is an absolutely harmonious graph, we can assign a subset of the numbers $\{0, 1, 2,\ q-1\}$ to the vertices of K_n in such a way that the edges receive each of the numbers a_0,a_1,...,a_{q-1} where $a_i = q-i\ or\ q+i$, $0 \le i \le q-1$. By Observation 3.10, $0, 1$, and 2 must be vertex labels. With vertices labeled $0, 1$, and 2, we have edges labeled a_{q-1}, a_{q-2} and a_{q-3}. To have an edge labeled a_{q-4} we must adjoin the vertex label 4. Had we adjoined the vertex label 3 to induce a_{q-4}, we would have two edges labeled a_{q-3}, namely, between 0 and 3, and between 1 and 2. Had we adjoined the vertex labels $q-1$

and $q-3$ to induce a_{q-4}, we would have three edges labeled a_1, namely, between $q-1$ and 0, between $q-1$ and 2, and between $q-3$ and 2. With vertices labeled $0, 1, 2,$and 4, we have edges labeled a_{q-1}, a_{q-2}, a_{q-3}, a_{q-4}, a_{q-5}, and a_{q-6}. Note that for K_4 with $q=6$, this gives the absolutely harmonious labeling. To have an edge labeled a_{q-7}, we must adjoin the vertex label 7; all the other choices are ruled out. With vertices labeled $0, 1, 2, 4$ and 7, we have edges labeled a_{q-1}, a_{q-2}, a_{q-3}, a_{q-4}, a_{q-5}, a_{q-6}, a_{q-7}, a_{q-8}, a_{q-9}, and a_{q-11}. There is now no way to obtain an edge labeled a_{q-10}, because each of the ways to induce a_{q-10} using two numbers contains at least one number that can not be assigned as vertex label. We may easily verify that the following boxed numbers are not possible choices as vertex labels:

$$\begin{array}{cc}
0 & \boxed{10} \\
1 & \boxed{9} \\
2 & \boxed{8} \\
\boxed{3} & 7 \\
4 & \boxed{6} \\
\boxed{q-1} & \boxed{q-9} \\
\boxed{q-2} & \boxed{q-8} \\
\boxed{q-3} & \boxed{q-7} \\
\boxed{q-4} & \boxed{q-6}
\end{array}$$

This contradiction proves the theorem. $\square$

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The Toroidal Crossing Number of $K_{4,n}$

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Abstract: In this paper, we study the crossing number of the complete bipartite graph $K_{4,n}$ in torus and obtain

$$cr_T(K_{4,n}) = \lfloor\frac{n}{4}\rfloor(2n - 4(1 + \lfloor\frac{n}{4}\rfloor)).$$

Key Words: Smarandache $\mathscr{P}$-drawing, crossing number, complete bipartite graph, torus.

AMS(2010): 05C10

§1. Introduction

A *complete bipartite graph* $K_{m,n}$ is a graph with vertex set $V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$, $|V_1| = m$ and $|V_2| = n$; and with edge set of all pairs of vertices with one element in V_1 and the other in V_2. The vertices in V_1 will be denoted by $b_i, b_j, b_k, \cdots$ and the vertices in V_2 will be denoted by $a_i, a_j, a_k, \cdots$.

A *drawing* is a mapping of a graph G into a surface. A *Smarandache $\mathscr{P}$-drawing* of a graph G for a graphical property $\mathscr{P}$ is such a good drawing of G on the plane with minimal intersections for its each subgraph $H \in \mathscr{P}$. A Smarandache $\mathscr{P}$-drawing is said to be *optimal* if $\mathscr{P} = G$ and it minimizes the number of crossings. Particularly, a drawing is *good* if it satisfies: (1) no two arcs which are incident with a common node have a common point; (2) no arc has a self-intersection; (3) no two arcs have more than one point in common; (4) no three arcs have a point in common. A common point of two arcs is called as a *crossing*. An *optimal drawing* in a given surface is a good drawing which has the smallest possible number of crossings. This number is the *crossing number* of the graph in the surface. We denote the crossing number of G in T, the torus, by $cr_T(G)$, a drawing of G in T by D. In this paper, we often speak of the nodes as vertices and the arcs as edges. For more graph terminologies and notations not mentioned here, you can refer to [1,3].

Garey and Johnson [2] stated that determining the crossing number of an arbitrary graph

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is NP-complete. In 1969, Guy and Jenkyns [4] proved that the crossing number of the complete bipartite graph $K_{3,n}$ in torus is $\lfloor\frac{(n-3)^2}{12}\rfloor$, and obtained the bounds on the crossing number of the complete bipartite graph $K_{m,n}$ in torus. In 1971, Kleitman [6] proved that the crossing number of the complete bipartite graph $K_{5,n}$ in plane is $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$ and the crossing number of the complete bipartite graph $K_{6,n}$ in plane is $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$. Later, Richter and Širáň [7] obtained the crossing number of the complete bipartite graph $K_{3,n}$ in an arbitrary surface. Recently, Ho [5] proved that the crossing number of the complete bipartite graph $K_{4,n}$ in real projective plane is $\lfloor\frac{n}{3}\rfloor(2n-3(1+\lfloor\frac{n}{3}\rfloor))$. In this paper, we obtain the crossing number of the complete bipartite graph $K_{4,n}$ in torus following.

Theorem 1 *The crossing number of the complete bipartite graph $K_{4,n}$ in torus is*

$$\mathrm{cr}_T(K_{4,n}) = \lfloor\frac{n}{4}\rfloor(2n-4(1+\lfloor\frac{n}{4}\rfloor)).$$

For convenience, let $f(n) = \lfloor\frac{n}{4}\rfloor(2n-4(1+\lfloor\frac{n}{4}\rfloor))$.

§2. Some Lemmas

In a drawing D of the complete bipartite $K_{m,n}$ in T, we denote by $cr_D(a_i, a_j)$ the number of crossings on edges one of which is incident with a vertex a_i and the other incident with a_j, and by $cr_D(a_i)$ the number of crossings on edges incident with a_i. Obviously,

$$cr_D(a_i) = \sum_{k=1}^{n} cr_D(a_i, a_k).$$

In every good drawing D, the *crossing number in D, $cr_T(D)$*, is

$$cr_T(D) = \sum_{i=1}^{n}\sum_{k=i+1}^{n} cr_D(a_i, a_k).$$

As $cr_D(a_i, a_i) = 0$ for all i, hence

$$cr_T(D) = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} cr_D(a_i, a_k) = \frac{1}{2}\sum_{i=1}^{n} cr_D(a_i). \tag{1}$$

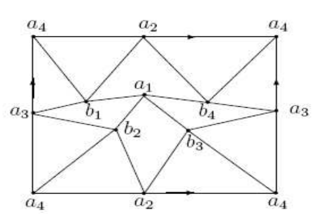

Fig.1

Fig.1. An optimal drawing of $K_{4,4}$ in T

Note that, in a crossing-free drawing of a connected subgraph of the complete bipartite graph $K_{m,n}$, every circuit has an even number of vertices, and in particular, every region into which the edges divide the surface is bounded by an even circuit. So, if F is the number of regions, E the number of edges and V the number of vertices, by the Eular's formula for T,

$$V - E + F \geq 0$$

$$F \geq E - V, \tag{2}$$

$$4F \leq 2E. \tag{3}$$

Suppose we have an optimal drawing of the complete bipartite graph $K_{m,n}$ in T, i.e., one with exactly $cr_T(K_{m,n})$ crossings. Then by deleting $cr_T(K_{m,n})$ edges, a crossing-free drawing will be obtained. From equations (2) and (3),

$$E - V = (mn - cr_T(K_{m,n})) - (m+n) \leq F \leq \frac{1}{2}E = \frac{1}{2}((mn - cr_T(K_{m,n})),$$

this implies

$$cr_T(K_{m,n}) \geq mn - 2(m+n). \tag{4}$$

In particular,

$$cr_T(K_{4,n}) \geq 2n - 8. \tag{5}$$

In Fig.1, it is a crossing-free drawing of the complete bipartite graph $K_{4,4}$ in T, hence

$$cr_T(K_{4,4}) = 0. \tag{6}$$

In paper [4], the following two lemmas can be find.

Lemma 1 *Let m, n, h be positive integers such that the complete bipartite graph $K_{m,h}$ embeds in T, then*

$$cr_T(K_{m,n}) \leq \frac{1}{2}\lfloor\frac{n}{h}\rfloor[2n - h(1 + \lfloor\frac{n}{h}\rfloor)]\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor.$$

Lemma 2 *If D is a good drawing of the complete bipartite graph $K_{m,n}$ in a surface Σ such that, for some $k < n$, some $K_{m,k}$ is optimally drawn in Σ, then*

$$cr_\Sigma(D) \geq cr_\Sigma(K_{m,k}) + (n-k)(cr_\Sigma(K_{m,k+1}) - cr_\Sigma(K_{m,k})) + cr_\Sigma(K_{m,n-k}).$$

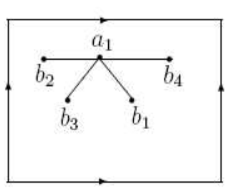

Fig.2

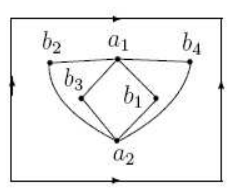

Fig.3

Lemma 3 *For $n \geq 4$, $cr_T(K_{4,n}) \leq f(n)$; especially, when $4 \leq n \leq 8$, $cr_T(K_{4,n}) = f(n)$.*

Proof As $cr_T(K_{4,4}) = 0$, by applying Lemma 1 with $m = h = 4$, then $cr_T(K_{4,n}) \leq f(n)$, $n \geq 4$. Especially, as $f(n) = 2n - 8$ for $4 \leq n \leq 8$, combining with equation (5), then $cr_T(K_{4,n}) = f(n)$ for $4 \leq n \leq 8$. □

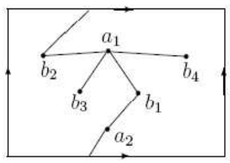

Fig.4

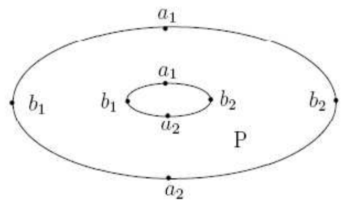

Fig.5

Lemma 4 *There is no good drawing D of $K_{4,5}$ in T such that*

(1) $cr_D(a_1, a_2) = cr_D(a_1, a_i) = cr_D(a_2, a_i) = 0$ *for* $3 \leq i \leq 5$;

(2) $cr_D(a_3, a_4) = cr_D(a_3, a_5) = cr_D(a_4, a_5) = 1$.

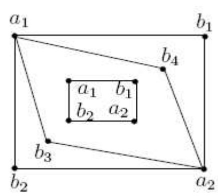

Fig.6(1)

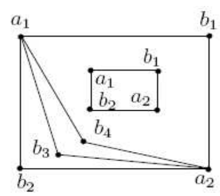

Fig.6(2)

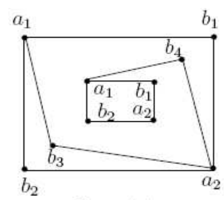

Fig.6(3)

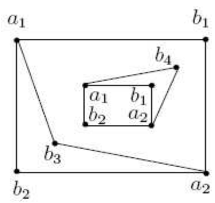

Fig.6(4)

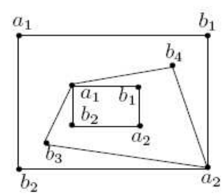

Fig.6(5)

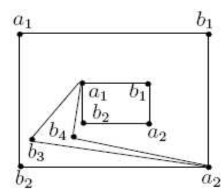

Fig.6(6)

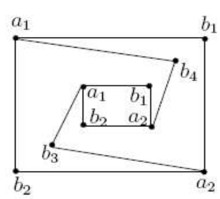

Fig.6(7)

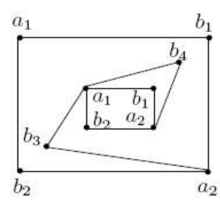

Fig.6(8)

Proof Note that T can be viewed as a rectangle with its opposite sides identified. As D is a good drawing, by deformation of the edges without changing the crossings and renaming the vertices if necessary, we can assume that the edges incident with a_1 are drawn as in Fig.2. Since $cr_D(a_1, a_2) = 0$, by deformation of edges without changing the crossings, we also assume that the edge a_2b_1 is drawn as in Fig.3. If the other three edges incident with a_2 are drawn without passing the sides of the rectangle (see Fig.3), then no matter which region a_3 is located, we have $cr_D(a_1, a_3) \geq 1$ or $cr_D(a_2, a_3) \geq 1$.

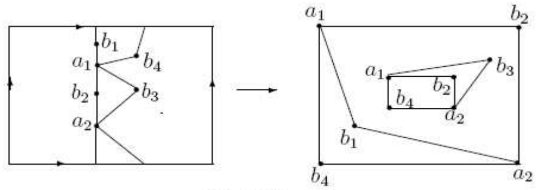

Fig.7(1)

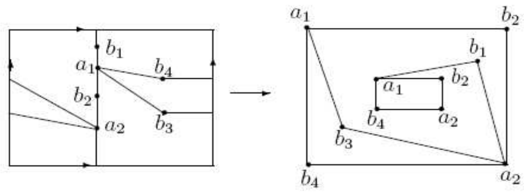

Fig.7(2)

So, there is at least one edge incident with a_2 which passes the sides of the rectangle. By deformation without changing the crossings and renaming the vertices if necessary, we assume that edge a_2b_2 passes the top and bottom sides of the rectangle only one time and is drawn as in Fig.4. Then we cut T along the circuit $a_1b_1a_2b_2a_1$ and obtain a surface which is homeomorphic to a ring in plane, denote by P, see Fig.5. Now, we put the vertices b_3, b_4 in P and use two rectangles to represent the outer and inner boundary which are both the circuit $a_1b_1a_2b_2a_1$.

As the vertices b_3 and b_4 are connected to a_1 and a_2 either in the outer or in the inner rectangle, which together presents 16 possibilities. In some cases, the four edges can either separate the two rectangles or not, implying up to 32 cases. Using symmetry, several cases are eliminated: without loss of generality, the vertex b_3 is connected to a_2 in the outer rectangle.

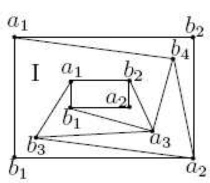

Fig.8(1)

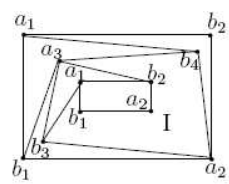

Fig.8(2)

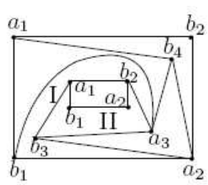

Fig.8(3)

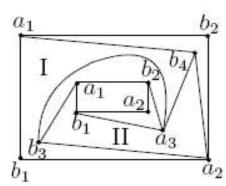

Fig.8(4)

First, assume that b_3 is also connected to a_1 in the outer rectangle. If b_4 is connected to both a_1 and a_2 in the outer rectangle, we obtain Fig.6(1) if the four edges separate the two rectangles, and Fig.6(2) if they do not. If b_4 is connected to a_1 in the inner rectangle and a_2 in the outer rectangle, we obtain Fig.6(3). If it is connected to a_1 in the outer rectangle and a_2 in the inner rectangle, then by relabeling a_1 and a_2, we obtain Fig.6(3). If b_4 is connected to both a_1 and a_2 in the inner rectangle, we obtain Fig.6(4).

Second, assume that b_3 is connected to a_1 in the inner rectangle. If b_4 is connected to both a_1 and a_2 in the outer rectangle, then by relabeling of b_3 and b_4, we obtain Fig.6(3). If b_4 is connected to a_1 in the inner rectangle and a_2 in the outer rectangle, we obtain Fig.6(5) if the four edges separate the two rectangles, and Fig.6(6) if they do not. If b_4 is connected to a_2 in the inner rectangle and a_1 in the outer rectangle, we obtain Fig.6(7). Finally, if b_4 is connected to both a_1 and a_2 in the inner rectangle, we obtain Fig.6(8).

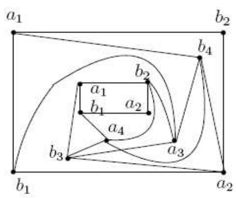

Fig.9(1)

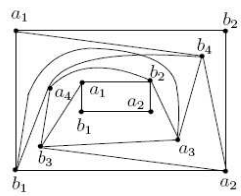

Fig.9(2)

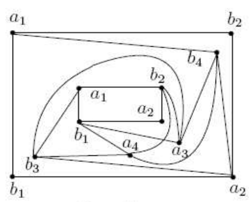

Fig.9(3)

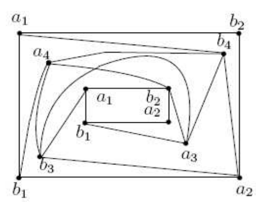

Fig.9(4)

Now, by drawing Fig.6(1) back into T and cut T along the circuit $a_1b_2a_2b_4a_1$, we obtain Fig.7(1); by drawing Fig.6(6) back into T and cut T along the circuit $a_1b_4a_2b_2a_1$, we obtain Fig.7(2). It is easy to find out that Fig.7(1) and Fig.6(4), Fig.7(2) and Fig.6(3) have the same structure if ignoring the labels of b. In Fig.6(8), by exchanging the inner and outer rectangles and the labels of b_3, b_4, we obtain Fig.6(3). In Fig.6(2), as each region has at most 3 vertices of $\{b_1, b_2, b_3, b_4\}$ on its boundary, we will have $cr_D(a_1, a_i) \geq 1$ or $cr_D(a_2, a_i) \geq 1$ for $i = 3, 4, 5$. So, we only need to consider the cases in Fig.6(3-5,7).

In Fig.6(3), since $cr_D(a_1, a_3) = cr_D(a_2, a_3) = 0$, we can draw the edges incident with a_3 in four different ways, see Fig.8(1-4). Furthermore, as $cr_D(a_1, a_4) = cr_D(a_2, a_4) = 0$ and $cr_D(a_3, a_4) = 1$, a_4 can only be putted in region I or II. In Fig.8(3-4), we can draw the edges incident with a_4 in four different ways, see Fig.9(1-4). In Fig.8(1-2), there are also four different ways to draw the edges incident with a_4, but they can be obtained by relabeling a_3 and a_4 in Fig.9((1-4). Then, we can see that no matter which region a_5 lies, we cannot have $cr_D(a_3, a_5) = cr_D(a_4, a_5) = 1$.

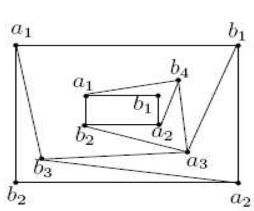

Fig.10(1)

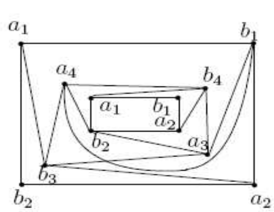

Fig.10(2)

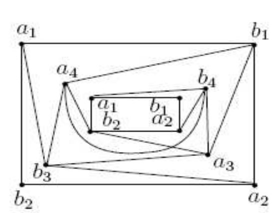

Fig.10(3)

In Fig.6(4), we have only one way to draw the edges incident with a_3, see Fig.10(1). Furthermore, we have two drawings of a_4 in Fig.10(1), see Fig.10(2-3). But, by observation, we

cannot have $cr_D(a_3, a_5) = cr_D(a_4, a_5) = 1$.

In Fig.6(5,7), no matter which regions a_3, a_4 locate, we will have $cr_D(a_3, a_4) \geq 2$ or $cr_D(a_3, a_4) = 0$. Now, the proof completes. □

§3. The proof of the Main Theorem

The proof of Theorem 1 is by induction on n. The base of the induction is $n \leq 8$ and has been obtained from Lemma 3. For $n \geq 9$, by Lemma 3, we only need to prove that $cr_T(K_{4,n}) \geq f(n)$. Let $n = 4q + r$ where $0 \leq r \leq 3$, and D be an optimal drawing of $K_{4,n}$ in T.

First, we assume that there exists a $K_{4,4}$ in D which is drawn without crossings. From Lemma 3, $cr_T(K_{4,5}) = 2$, and by the inductive assumption, $cr_T(K_{4,n-4}) = f(n-4)$. Hence, by applying Lemma 2 with $m = k = 4$,

$$\begin{aligned} cr_T(D) &\geq 2(n-4) + f(n-4) = 2(n-4) + \lfloor \frac{n-4}{4} \rfloor (2(n-4) - 4(1 + \lfloor \frac{n-4}{4} \rfloor)) \\ &= 8q + 2r - 8 + (q-1)(4q + 2r - 8) = 4q^2 + 2qr - 4q, \end{aligned}$$

which is $f(n)$, since

$$f(n) = \lfloor \frac{n}{4} \rfloor (2n - 4(1 + \lfloor \frac{n}{4} \rfloor)) = q(8q + 2r - 4(1+q)) = 4q^2 + 2qr - 4q. \tag{7}$$

Second, we assume that every $K_{4,4}$ in D is drawn with at least one crossings. Clearly, $K_{4,n}$ contains n subgraphs $K_{4,n-1}$, each contains at least $f(n-1)$ crossings by the inductive hypothesis. As each crossing will be counted $n-2$ times, hence

$$cr_T(D) \geq \frac{n}{n-2} cr_T(K_{4,n-1}) = \frac{n}{n-2} f(n-1). \tag{8}$$

From equation (7),

$$f(n) = \begin{cases} q(4q-4), & \text{for } n = 4q, \\ q(4q-2), & \text{for } n = 4q+1, \\ 4q^2, & \text{for } n = 4q+2, \\ q(4q+2), & \text{for } n = 4q+3. \end{cases}$$

Combining this with equation (8),

$$cr_T(D) \geq \begin{cases} q(4q-4), & \text{for } n = 4q, \\ q(4q-2) - 1 - \frac{2q+1}{4q-1}, & \text{for } n = 4q+1, \\ 4q^2 - 1, & \text{for } n = 4q+2, \\ q(4q+2) - \frac{2q}{4q+1}, & \text{for } n = 4q+3. \end{cases}$$

As $n \geq 9$, namely $q \geq 2$, and the crossing number is an integer, thus, when $n = 4q$ or $4q+3$,

$$cr_T(K_{4,n}) = cr_T(D) \geq f(n);$$

when $n = 4q+1$ or $4q+2$,

$$cr_T(K_{4,n}) = cr_T(D) \geq f(n) - 1.$$

Therefore, only the two cases $n = 4q+1$ and $n = 4q+2$ are needed considering. In the following, we assume that $cr_T(K_{4,n}) = cr_T(D) = f(n)-1$ for $n = 4q+1$ or $4q+2$, and denote the drawing of $K_{4,n-1}$ obtained by deleting the vertex a_i of $K_{4,n}$ in D by $D - \{a_i\}$.

Case 1. $n = 4q + 1$.

By the inductive assumption,

$$cr_T(D - \{a_i\}) \geq f(4q),\ 1 \leq i \leq 4q + 1.$$

As $cr_T(D) = f(4q + 1) - 1 = 4q^2 - 2q - 1$, then

$$cr_D(a_i) = cr_T(D) - cr_T(D - \{a_i\}) \leq f(4q + 1) - 1 - f(4q) = 2q - 1,\ 1 \leq i \leq 4q + 1.$$

Let x be the number of a_i such that $cr_D(a_i) = 2q - 1$, y be the number of a_i such that $cr_D(a_i) = 2q - 2$, thus, the number of a_i such that $cr_D(a_i) \leq 2q - 3$ is $4q + 1 - (x + y)$. By equation(1), it holds

$$\begin{aligned}(2q-1)x &+ (2q-2)y + (4q+1-x-y)(2q-3) \geq 2cr_T(D) = 8q^2 - 4q - 2\\ 2x + y &\geq 6q + 1.\end{aligned}$$

As $x + y \leq 4q + 1$, then $x \geq 2q$. Without loss of generality, by renaming the vertices, suppose that $cr_D(a_i) = 2q - 1$ for $i \leq x$.

Case 1.1 There exists a pair of (i, j), $1 \leq i < j \leq x$, such that $cr_D(a_i, a_j) = 0$. Denote the drawing of the graph $K_{4,4q-1}$ obtained by deleting the vertices a_i, a_j of the graph $K_{4,4q+1}$ in D by $D - \{a_i, a_j\}$. Then,

$$cr_T(D - \{a_i, a_j\}) = f(4q + 1) - 1 - 2(2q - 1) = 4q^2 - 6q + 1.$$

But this contradicts the inductive assumption that $cr_T(K_{4,4q-1}) = f(4q - 1) = 4q^2 - 6q + 2$.

Case 1.2 For every (i, j), $1 \leq i < j \leq x$, $cr_D(a_i, a_j) \geq 1$. As $cr_D(a_i) = 2q - 1$, obviously, $x = 2q$ and

$$cr_D(a_i, a_j) = 1,\ 1 \leq i < j \leq 2q,\ cr_D(a_i, a_h) = 0,\ 1 \leq i \leq 2q < h \leq 4q + 1.$$

Furthermore, as $x + y \leq 4q + 1$ and $2x + y \geq 6q + 1$, then $y = 2q + 1$. By the definition of y, there exist a_h, a_k, where $2q + 1 \leq h < k \leq 4q + 1$, such that $cr_D(a_h, a_k) = 0$. Now, we obtain a drawing of $K_{4,5}$ in T with vertices a_h, a_k, a_1, a_2, a_3 such that $cr_D(a_h, a_k) = cr_D(a_h, a_i) = cr_D(a_k, a_i) = 0$ $(1 \leq i \leq 3)$ and $cr_D(a_1, a_2) = cr_D(a_1, a_3) = cr_D(a_2, a_3) = 1$. Contradicts to Lemma 4.

Combining the above two subcases, we have $cr_T(K_{4,4q+1}) = f(4q + 1) = q(4q - 2)$.

Case 2. $n = 4q + 2$.

By the inductive assumption,

$$cr_T(D - \{a_i\}) \geq f(4q + 1) = q(4q - 2),\ 1 \leq i \leq 4q + 2.$$

As $cr_T(D) = f(4q+2) - 1 = 4q^2 - 1$, thus

$$cr_D(a_i) = cr_T(D) - cr_T(D - \{a_i\}) \leq (f(4q+2) - 1) - f(4q+1) = 2q - 1.$$

Let t be the number of a_i such that $cr_D(a_i) = 2q - 1$, then there are $(4q + 2 - t)$ vertices a_i such that $cr_D(a_i) \leq 2q - 2$. From equation (1),

$$\begin{aligned}(2q-1)t &+ (2q-2)(4q+2-t) \geq 2cr_T(D) = 8q^2 - 2\\ t &\geq 4q+2.\end{aligned}$$

As $t \leq n = 4q + 2$, hence, $t = 4q + 2$, this implies that $cr_D(a_i) = 2q - 1$ $(1 \leq i \leq 4q + 2)$.

If there exists a pair of (i, j), $1 \leq i < j \leq 4q + 2$, such that $cr_D(a_i, a_j) \geq 3$, then,

$$cr_T(D - \{a_i\}) = cr_T(D) - cr_D(a_i) = 4q^2 - 1 - (2q - 1) = 4q^2 - 2q,$$

and

$$cr_{(D-\{a_i\})}(a_j) = cr_D(a_j) - cr_D(a_i, a_j) \leq 2q - 1 - 3 = 2q - 4.$$

Now, by putting a new vertex $a_i^{'}$ near the vertex a_j in $D - \{a_i\}$ and drawing the edges $a_i^{'}b_k(1 \leq k \leq 4)$ nearly to a_jb_k, a new drawing of $K_{4,4q+2}$ in T is obtained, denoted by $D^{'}$. Clearly,

$$cr_{D'}(a_i^{'}, a_j) = 2 \text{ and } cr_{D'}(a_i^{'}, a_h) = cr_{D-\{a_i\}}(a_j, a_h),\ h \neq j.$$

Thus,

$$cr_T(D^{'}) = cr_T(D - \{a_i\}) + 2 + cr_{(D-\{a_i\})}(a_j) \leq 4q^2 - 2.$$

But, this contradicts to the hypothesis that $cr_T(K_{4,4q+2}) \geq 4q^2 - 1$.

Therefore, for $1 \leq i < j \leq 4q + 2$, $cr_D(a_i, a_j) \leq 2$. For each a_i, $1 \leq i \leq 4q + 2$, let

$$S_0^{(i)} = \{a_j \,|\, cr_D(a_i, a_j) = 0,\ j \neq i\}, \quad S_{\geq 1}^{(i)} = \{a_j \,|\, cr_D(a_i, a_j) \geq 1\},$$

$$S_1^{(i)} = \{a_j \,|\, cr_D(a_i, a_j) = 1\}, \qquad S_2^{(i)} = \{a_j \,|\, cr_D(a_i, a_j) = 2\}.$$

As $cr_D(a_i, a_j) \leq 2$, $cr_D(a_i) = 2q - 1$ is odd, then, for $1 \leq i \leq 4q + 2$,

$$\emptyset \neq S_1^{(i)} \subseteq S_{\geq 1}^{(i)}, \quad |S_1^{(i)}| + |S_2^{(i)}| = |S_{\geq 1}^{(i)}|, \quad |S_{\geq 1}^{(i)}| = 2q - 1 - |S_2^{(i)}|. \tag{9}$$

Furthermore, since $q \geq 2$,

$$|S_0^{(i)}| = 4q + 2 - 1 - |S_{\geq 1}^{(i)}| = 2q + 2 + |S_2^{(i)}| \geq 6.$$

For $1 \leq i < j \leq 4q + 2$, clearly,

$$S_0^{(i)} \cup S_{\geq 1}^{(i)} \cup \{a_i\} = S_0^{(j)} \cup S_{\geq 1}^{(j)} \cup \{a_j\}.$$

If $cr_D(a_i, a_j) = 0$ and $S_{\geq 1}^{(i)} \cap S_{\geq 1}^{(j)} = \emptyset$, then, the above equation implies that

$$S_{\geq 1}^{(i)} \subseteq S_0^{(j)} \quad \text{and} \quad S_{\geq 1}^{(j)} \subseteq S_0^{(i)}. \tag{10}$$

Without loss of generality, let

$$|S_2^{(1)}| = \max\{|S_2^{(i)}| \mid 1 \le i \le 4q+2\}, \quad |S_2^{(2)}| = \max\{|S_2^{(j)}| \mid a_j \in S_0^{(1)}\}.$$

For $3 \le i \le 4q+2$, if $a_i \notin S_{\ge 1}^{(1)} \cup S_{\ge 1}^{(2)}$, then $a_i \in S_0^{(1)} \cap S_0^{(2)}$. This means that

$$|S_0^{(1)} \cap S_0^{(2)}| = 4q - |S_{\ge 1}^{(1)} \cup S_{\ge 1}^{(2)}| = 4q - |S_{\ge 1}^{(1)}| - |S_{\ge 1}^{(2)}| + |S_{\ge 1}^{(1)} \cap S_{\ge 1}^{(2)}|.$$

From equation (9), then

$$|S_0^{(1)} \cap S_0^{(2)}| = 2 + |S_2^{(1)}| + |S_2^{(2)}| + |S_{\ge 1}^{(1)} \cap S_{\ge 1}^{(2)}|. \tag{11}$$

With these notations, it is obvious that $|S_2^{(1)}| \ge |S_2^{(2)}|$ and $cr_D(a_1, a_2) = 0$. In the following, the discussions are divided into two subcases according to $S_{\ge 1}^{(1)} \cap S_{\ge 1}^{(2)} = \emptyset$ or not.

Case 2.1 $S_{\ge 1}^{(1)} \cap S_{\ge 1}^{(2)} \ne \emptyset$. Let $|S_{\ge 1}^{(1)} \cap S_{\ge 1}^{(2)}| = \alpha \ge 1$, from equation (11),

$$|S_0^{(1)} \cap S_0^{(2)}| = 2 + |S_2^{(1)}| + |S_2^{(2)}| + \alpha.$$

First, we choose a vertex from $S_0^{(1)} \cap S_0^{(2)}$, without loss of generality, denoted by a_3. By the assumption that every $K_{4,4}$ in D is drawn with at least one crossings, hence $cr_D(a_3, a_i) \ge 1$ for all $a_i \in S_0^{(1)} \cap S_0^{(2)}$, $a_i \ne a_3$. Let $U = \{a_i \mid cr_D(a_3, a_i) = 1,\ a_i \in S_0^{(1)} \cap S_0^{(2)}\}$. Since $a_3 \in S_0^{(1)}$ and $|S_2^{(2)}| = \max\{|S_2^{(j)}| \mid a_j \in S_0^{(1)}\}$, then $|S_2^{(3)}| \le |S_2^{(2)}|$ and

$$|U| \ge |S_0^{(1)} \cap S_0^{(2)}| - 1 - |S_2^{(3)}| \ge 1 + |S_2^{(1)}| + \alpha.$$

Second, we choose a vertex from U, denoted by a_4. By the assumption that every $K_{4,4}$ in D is drawn with at least one crossings, $cr_D(a_4, a_i) \ge 1$ for all $a_i \in U$, $a_i \ne a_4$. As $|S_2^{(4)}| \le |S_2^{(1)}|$(for $|S_2^{(1)}| = \max\{|S_2^{(i)}| \mid 1 \le i \le 4q+2\}$), thus $|U \setminus S_2^{(4)}| \ge \alpha \ge 1$ and there exists one vertex in U, denoted by a_5, such that $cr_D(a_4, a_5) = 1$. Now, we have a drawing of $K_{4,5}$ in T with vertices a_1, a_2, a_3, a_4, a_5 such that $cr_D(a_1, a_2) = cr_D(a_1, a_k) = cr_D(a_2, a_k) = 0$ for $3 \le k \le 5$ and $cr_D(a_3, a_4) = cr_D(a_3, a_5) = cr_D(a_4, a_5) = 1$. But, this contradicts to Lemma 4.

Case 2.2 $S_{\ge 1}^{(1)} \cap S_{\ge 1}^{(2)} = \emptyset$. From equation (11),

$$|S_0^{(1)} \cap S_0^{(2)}| = 2 + |S_2^{(1)}| + |S_2^{(2)}|.$$

We choose a vertex from $S_0^{(1)} \cap S_0^{(2)}$, also denoted by a_3. By the same discussion as in case 2.1, we have $cr_D(a_3, a_i) \ge 1$ for all $a_i \in S_0^{(1)} \cap S_0^{(2)}$, $a_i \ne a_3$. Let $\Lambda = \{a_i \mid cr_D(a_3, a_i) = 2,\ a_i \in S_0^{(1)} \cap S_0^{(2)}\}$, $\Phi = \{a_i \mid cr_D(a_3, a_i) = 1,\ a_i \in S_0^{(1)} \cap S_0^{(2)}\}$. As $a_3 \in S_0^{(1)}$, $|S_2^{(2)}| = \max\{|S_2^{(j)}| \mid a_j \in S_0^{(1)}\}$ and $|S_2^{(1)}| = \max\{|S_2^{(i)}| \mid 1 \le i \le 4q+2\}$, then

$$\Lambda \subseteq S_2^{(3)}, \quad |\Lambda| \le |S_2^{(3)}| \le |S_2^{(2)}| \le |S_2^{(1)}|, \tag{12}$$

and

$$|\Phi| = |S_0^{(1)} \cap S_0^{(2)}| - 1 - |\Lambda| = 1 + |S_2^{(1)}| + |S_2^{(2)}| - |\Lambda| \tag{13}$$

If there are two vertices in Φ, denoted by a_4, a_5, such that $cr_D(a_4, a_5) = 1$. Then we also have a drawing of $K_{4,5}$ with vertices a_1, a_2, a_3, a_4, a_5 which will contradict to Lemma 4. Hence,

for all $a_i, a_j \in \Phi(a_i \neq a_j)$, $cr_D(a_i, a_j) \neq 1$, this implies that $cr_D(a_i, a_j) = 2$ since $cr_D(a_i, a_j)$ cannot be zero (otherwise there exists $K_{4,4}$ in D drawn with no crossings), and

$$|S_2^{(i)}| \geq |\Phi| - 1.$$

Furthermore, if $|\Lambda| < |S_2^{(2)}|$, by equation(13), $|\Phi| > 1 + |S_2^{(1)}|$, and for each $a_i \in \Phi$,

$$|S_2^{(i)}| \geq |\Phi| - 1 > |S_2^{(1)}|.$$

This contradicts the maximum of $|S_2^{(1)}|$. Thus,

$$|\Lambda| = |S_2^{(2)}|, \ |\Phi| = 1 + |S_2^{(1)}|,$$

and for each $a_i \in \Phi$,

$$|S_2^{(i)}| \geq |\Phi| - 1 = |S_2^{(1)}|.$$

As $|S_2^{(i)}| \leq |S_2^{(2)}| \leq |S_2^{(1)}|$, combining equation (12),

$$|S_2^{(1)}| = |S_2^{(2)}| = |S_2^{(3)}| = |S_2^{(i)}|, \tag{14}$$

and

$$S_2^{(3)} = \Lambda \subseteq S_0^{(1)} \cap S_0^{(2)}.$$

Combining equations (14) and (9), for each $a_i \in \Phi$,

$$|S_{\geq 1}^{(1)}| = |S_{\geq 1}^{(2)}| = |S_{\geq 1}^{(3)}| = |S_{\geq 1}^{(i)}|,$$

and

$$|S_1^{(1)}| = |S_1^{(2)}| = |S_1^{(3)}| = |S_1^{(i)}|.$$

As $|\Phi| = 1 + |S_2^{(1)}| + |S_2^{(2)}| - |\Lambda| \geq 1$, we choose a vertex from Φ and denote it by a_4.

If there exists a pair of (i, j), $i \in \{1, 2\}$ and $j \in \{3, 4\}$, such that $S_{\geq 1}^{(i)} \cap S_{\geq 1}^{(j)} \neq \emptyset$, by replacing $S_{\geq 1}^{(1)} \cap S_{\geq 1}^{(2)} \neq \emptyset$ with $S_{\geq 1}^{(i)} \cap S_{\geq 1}^{(j)} \neq \emptyset$ in case 2.1, as $a_j \in S_0^{(1)} \cap S_0^{(2)} (j = 3, 4)$ and $|S_2^{(i)}| = |S_2^{(j)}| = \max\{|S_2^{(k)}| \mid 1 \leq k \leq 4q + 2\}$, we also can obtain a contradiction to Lemma 4.

So, for every (i, j), $i \in \{1, 2\}$ and $j \in \{3, 4\}$, $S_{\geq 1}^{(i)} \cap S_{\geq 1}^{(j)} = \emptyset$. As $S_{\geq 1}^{(1)} \cap S_{\geq 1}^{(2)} = \emptyset$ and $cr_D(a_i, a_j) = cr_D(a_1, a_2) = 0$, combining equations (9) and (10), then

$$\emptyset \neq S_1^{(1)} \subseteq S_{\geq 1}^{(1)} \subseteq S_0^{(2)} \cap S_0^{(3)} \cap S_0^{(4)} \quad \text{and} \quad \emptyset \neq S_1^{(2)} \subseteq S_{\geq 1}^{(2)} \subseteq S_0^{(1)} \cap S_0^{(3)} \cap S_0^{(4)}.$$

Since $S_1^{(1)} \neq \emptyset$, there exists a vertex, denoted by a_5, such that $a_5 \in S_1^{(1)} \subseteq S_0^{(2)} \cap S_0^{(3)} \cap S_0^{(4)}$. This implies that

$$cr_D(a_1, a_5) = 1 \text{ and } cr_D(a_2, a_5) = cr_D(a_3, a_5) = cr_D(a_4, a_5) = 0.$$

As $S_{\geq 1}^{(2)} \cap S_{\geq 1}^{(3)} = \emptyset$, $|S_2^{(1)}| = |S_2^{(2)}| = |S_2^{(3)}|$, $cr_D(a_2, a_3) = 0$ and $a_5 \subseteq S_0^{(2)} \cap S_0^{(3)}$, by replacing $S_{\geq 1}^{(1)} \cap S_{\geq 1}^{(2)} = \emptyset$ with $S_{\geq 1}^{(2)} \cap S_{\geq 1}^{(3)} = \emptyset$ and replacing a_3 with a_5 in the beginning part of Case 2.2, we also can obtain that $|S_1^{(5)}| = |S_1^{(2)}| = |S_1^{(3)}|$ and $S_2^{(5)} \subseteq S_0^{(2)} \cap S_0^{(3)}$. This means that, for any vertex $a_k \in S_1^{(2)}$,

$$cr_D(a_5, a_k) \leq 1. \tag{15}$$

As $S_1^{(2)} \neq \emptyset$, there exists one vertex in $S_1^{(2)}$, denoted by a_6, such that $cr_D(a_5, a_6) = 0$. Otherwise, from equation (15) and $cr_D(a_1, a_5) = 1$, $S_1^{(2)} \cup \{a_1\} \subseteq S_1^{(5)}$. As $a_1 \notin S_1^{(2)}$, then $|S_1^{(5)}| \geq |S_1^{(2)}| + 1$, which contradicts to $|S_1^{(5)}| = |S_1^{(2)}| = |S_1^{(3)}|$. Furthermore, as $a_6 \in S_1^{(2)} \subseteq S_0^{(1)} \cap S_0^{(3)} \cap S_0^{(4)}$, we also have

$$cr_D(a_2, a_6) = 1 \text{ and } cr_D(a_1, a_6) = cr_D(a_3, a_6) = cr_D(a_4, a_6) = 0.$$

Hence, we obtain a good drawing of $K_{4,6}$ in T, denoted by D', with

$$cr_{D'}(a_i) = \sum_{j=1}^{6} cr_D(a_i, a_j) = 1, \ 1 \leq i \leq 6,$$

and

$$cr_T(K_{4,6}) \leq cr_T(D') = \frac{1}{2} \sum_{i=1}^{6} cr_{D'}(a_i) = 3.$$

This contradicts to Lemma 3. Thus, $cr_T(K_{4,4q+2}) = cr_T(D) = f(4q+2) = 4q^2$. $\square$

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On Pathos Semitotal and Total Block Graph of a Tree

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Abstract: In this communications, the concept of pathos semitotal and total block graph of a graph is introduced. Its study is concentrated only on trees. We present a characterization of those graphs whose pathos semitotal block graphs are planar, maximal outer planar, non-minimally non-outer planar, non-Eulerian and hamiltonian. Also, we present a characterization of graphs whose pathos total block graphs are planar, maximal outer planar, minimally non-outer planar, non-Eulerian, hamiltonian and graphs with crossing number one.

Key Words: Pathos, path number, Smarandachely block graph, semitotal block graph, Total block graph, pathos semitotal graph, pathos total block graph, pathos length, pathos point, inner point number.

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§1. Introduction

The concept of pathos of a graph G was introduced by Harary [2], as a collection of minimum number of line disjoint open paths whose union is G. The path number of a graph G is the number of paths in pathos. A new concept of a graph valued functions called the semitotal and total block graph of a graph was introduced by Kulli [6]. For a graph $G(p, q)$ if $B = \{u_1, u_2, u_3, \ldots, u_r; r \geq 2\}$ is a block of G, then we say that point u_1 and block B are incident with each other, as are u_2 and B and so on. If two distinct blocks B_1 and B_2 are incident with a common cut point, then they are adjacent blocks. The points and blocks of a graph are called its members. A *Smarandachely block graph* $T_S^V(G)$ for a subset $V \subset V(G)$ is such a graph with vertices $V \cup \mathcal{B}$ in which two points are adjacent if and only if the corresponding members of G are adjacent in $\langle V \rangle_G$ or incident in G, where $\mathcal{B}$ is the set of blocks of G. The semitotal block graph of a graph G denoted by $T_b(G)$ is defined as the graph whose point set is the union of set of points, set of blocks of G in which two points are adjacent if and only if members of G are incident, thus a Smarandachely block graph with $V = \emptyset$. The total block graph of a graph

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G denoted by $T_B(G)$ is defined as the graph whose point set is the union of set of points, set of blocks of G in which two points are adjacent if and only if the corresponding members of G are adjacent or incident, i.e., a Smarandachely block graph with $V = V(G)$. Stanton [11] and Harary [3] have calculated the path number for certain classes of graphs like trees and complete graphs.

All undefined terminology will conform with that in Harary [1]. All graphs considered here are finite, undirected and without loops or multiple lines.

The pathos semitotal block graph of a tree T denoted by $P_{T_B}(T)$ is defined as the graph whose point set is the union of set of points, set of blocks and the set of path of pathos of T in which two points are adjacent if and only if the corresponding members of G are incident and the lines lie on the corresponding path P_i of pathos. Since the system of pathos for a tree is not unique, the corresponding pathos semitotal and pathos total block graph of a tree T is also not unique.

In Fig.1, a tree T, its semitotal block graph $T_b(T)$ and their pathos semi total block $P_{T_b}(T)$ graph are shown. In Fig. 2, a tree T, its semitotal block graph $T_b(T)$ and their pathos total block $P_{T_B}(T)$ graph are shown.

The line degree of a line uv in a tree T, pathos length, pathos point in T was defined by Muddebihal [10]. If G is planar, the inner point number $i(G)$ of a graph G is the minimum number of points not belonging to the boundary of the exterior region in any embedding of G in the plane. A graph G is said to be minimally nonouterplanar if $i(G) = 1$, as was given by Kulli [4].

We need the following results to prove further results.

Theorem [A][Ref.6] *If G is connected graph with p points and q lines and if b_i is the number of blocks to which v_i belongs in G, then the semitotal block graph $T_b(G)$ has $\left(\sum_{i=1}^{p} b_i\right)+1$, points and $q + \left(\sum_{i=1}^{p} b_i\right)$ lines.*

Theorem [B][Ref.6] *If G is connected graph with p points and q lines and if b_i is the number of blocks to which v_i belongs in G, then the total block graph $T_B(G)$ has $\left(\sum_{i=1}^{p} b_i\right) + 1$, points and $q + \sum_{i=1}^{p} \begin{pmatrix} b_i + 1 \\ 2 \end{pmatrix}$ lines.*

Theorem [C][Ref.8] *The total block graph $T_B(G)$ of a graph G is planar if and only if G is outerplanar and every cutpoint of G lies on atmost three blocks.*

Theorem [D] [Ref.7] *The total block graph $T_B(G)$ of a connected graph G is minimally nonouter planar if and only if,*

(1) *G is a cycle, or*

(2) *G is a path P of length $n \geq 2$, together with a point which is adjacent to any two adjacent points of P.*

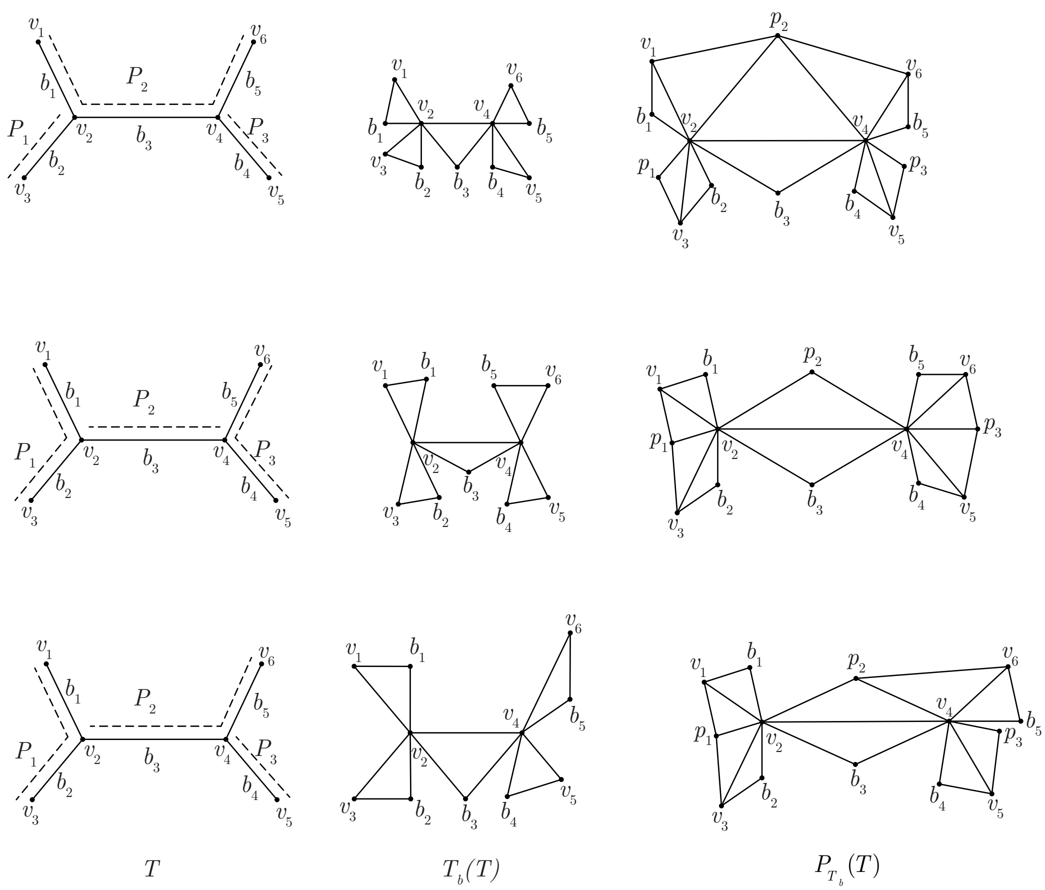


Figure 1:

Theorem [E][Ref.9] *The total block graph $T_B(G)$ of a graph G crossing number 1 if and only if*

(1) *G is outer planar and every cut point in G lies on at most 4 blocks and G has a unique cut point which lies on 4 blocks, or*

(2) *G is minimally non-outer planar, every cut point of G lies on at most 3 blocks and exactly one block of G is theta-minimally non-outer planar.*

Corollary [A][Ref.1] *Every nontrivial tree contains at least two end points.*

Theorem [F][Ref.1] *Every maximal outerplanar graph G with p points has $(2p-3)$ lines.*

Theorem [G][Ref.5] *A graph G is a non empty path if and only if it is connected graph with $p \geq 2$ points and $\sum\limits_{i=1}^{p} {d_i}^2 - 4p + 6 = 0$.*

§2. Pathos Semitotal Block Graph of a Tree

We start with a few preliminary results.

Remark 1 *The number of blocks in pathos semitotal block graph of $P_{T_b}(T)$ of a tree T is equal to the number of pathos in T.*

Remark 2 *If the degree of a pathos point in pathos semi total block graph $P_{T_b}(T)$ of a tree T is n, then the pathos length of the corresponding path P_i of pathos in T is $n-1$.*

Kulli [6] developed the new concept in graph valued functions i.e., semi total and total block graph of a graph. In this article the number of points and lines of a semi total block graph of a graph has been expressed in terms of blocks of G. Now using this we have a modified theorem as shown below in which we have expressed the number of points and lines in terms of lines and degrees of the points of G which is a tree.

Theorem 1 *For any (p,q) tree T, the semitotal block graph $T_b(T)$ has $(2q+1)$ points and $3q$ lines.*

Proof By Theorem [A], the number of points in $T_b(G)$ is $\left(\sum\limits_{i=1}^{p} b_i\right)+1$, where b_i are the number of blocks in T to which the points v_i belongs in G. Since $\sum b_i = 2q$, for G is a tree. Thus the number of points in $T_b(G) = 2q+1$. Also, by Theorem [A] the number of lines in $T_b(G)$ are $q + \left(\sum\limits_{i=1}^{b} b_i\right)$, since $\sum b_i = 2q$ for G is a tree. Thus the number of lines in $T_b(G)$ is $q+2q = 3q$. □

In the following theorem we obtain the number of points and lines in $P_{T_b}(T)$.

Theorem 2 *For any non trivial tree T, the pathos semitotal block graph of a tree T, whose points have degree d_i, then the number of points in $P_{T_b}(T)$ are $(2q+k+1)$ and the number of lines are $\left(2q+2+\frac{1}{2}\sum\limits_{i=1}^{p} d_i^2\right)$, where k is the path number.*

Proof By Theorem 1, the number of points in $T_b(T)$ are $2q+1$, and by definition of $P_{T_b}(T)$, the number of points in (2q+k+1), where k is the path number. Also by Theorem 1, the number of lines in $T_b(T)$ are $3q$. The number of lines in $P_{T_b}(T)$ is the sum of lines in $T_b(T)$ and the number of lines which lie on the points of pathos of T which are to $\left(-q+2+\frac{1}{2}\sum\limits_{i=1}^{p} d_i^2\right)$. Thus the number of lines in is equal to $\left(3q+(-q+2+\frac{1}{2}\sum\limits_{i=1}^{p} d_i^2)\right) = \left(2q+2+\frac{1}{2}\sum\limits_{i=1}^{p} d_i^2\right)$.

§2. Planar Pathos Semitotal Block Graphs

A criterion for pathos semi total block graph to be planar is presented in our next theorem.

Theorem 3 *For any non trivial tree T, the pathos semitotal block graph $P_{T_b}(T)$ of a tree T is planar.*

Proof Let T be a non trivial tree, then in $T_b(T)$ each block is a triangle. We have the following cases.

Case 1 Suppose G is a path, $G = P_n : u_1, u_2, u_3, \ldots, u_n, n > 1$. Further, $V\left[T_b\left(T\right)\right] =$

$\{u_1, u_2, u_3, \ldots, u_n, b_1, b_2, b_3, \ldots, b_{n-1}\}$, where $b_1, b_2, b_3, \ldots, b_{n-1}$ are the corresponding block points. In pathos semi total block graph $P_{T_b}(T)$ of a tree T, $\{u_1b_1u_2w, u_2b_2u_3w, u_3b_3u_4w, \ldots, u_{n-1}b_{n-1}u_nw\} \in V[P_{T_b}(T)]$, each set $\{u_{n-1}b_{n-1}u_nw\}$ forms an induced subgraph as $K_4 - x$. Hence one can easily verify that $P_{T_b}(T)$ is planar.

Case 2 Suppose G is not a path. Then $V[T_b(G)] = \{u_1, u_2, u_3, \ldots, u_n, b_1, b_2, b_3, \ldots, b_{n-1}\}$ and $w_1, w_2, w_3, \ldots, w_k$ be the pathos points. Since $u_{n-1}u_n$ is a line and $u_{n-1}u_n = b_{n-1} \in V[T_b(G)]$. Then in $P_{T_b}(G)$ the set $\{u_{n-1}b_{n-1}u_nw\}$ $\forall\, n > 1$, forms $K_4 - x$ as an induced subgraphs. Hence $P_{T_b}(G)$ is planar. $\square$

Further we develop the maximal outer planarity of $P_{T_b}(G)$ in the following theorem.

Theorem 4 *For any non trivial tree T, the pathos semitotal block graph $P_{T_b}(T)$ of a tree T is maximal outer planar if and only if T is a path.*

Proof Suppose $P_{T_b}(T)$ is maximal outer planar. Then $P_{T_b}(T)$ is connected. Hence T is connected. If $P_{T_b}(T)$, is $K_4 - x$, then obviously T is k_2.

Let T be any connected tree with $p \geq 2$, q lines b_i blocks and path number k, then clearly $P_{T_b}(T)$ has $(2q + k + 1)$ points and $\left(2q + 2 + \frac{1}{2}\sum_{i=1}^{p} d_i^2\right)$ lines. Since $P_{T_b}(T)$ is maximal outer planar, by Theorem [F], it has $[2(2q + k + 1) - 3]$ lines. Hence,

$$2 + 2q + \frac{1}{2}\sum_{i=1}^{p} d_i^2 = 2\,(2\mathrm{q} + \mathrm{k} + 1) - 3 = \ 4\mathrm{q} + 2\mathrm{k} + 2 - 3 = \ 4\mathrm{q} + 2\mathrm{q} - 1$$

$$\frac{1}{2}\sum_{i=1}^{p} d_i^2 = 2\mathrm{q} + 2\mathrm{k} - 3$$

$$\sum_{i=1}^{p} d_i^2 = 4\mathrm{q} + 4\mathrm{k} - 6$$

$$\sum_{i=1}^{p} d_i^2 = \ 4\,(\mathrm{p} - 1) \ \ + 4\mathrm{k} - 6$$

$$\sum_{i=1}^{p} d_i^2 = 4\mathrm{p} + 4\mathrm{k} - 10.$$

But for a path, $k = 1$.

$$\sum_{i=1}^{p} d_i^2 = 4\mathrm{p} + 4\,(1) - 10 = \ 4\mathrm{p} - 6$$

$$\sum_{i=1}^{p} d_i^2 - 4\mathrm{p} + 6 = 0.$$

By Theorem [G], it follows that T is a non empty path. Thus necessity is proved.

For sufficiency, suppose T is a non empty path. We prove that $P_{T_b}(T)$ is maximal outer planar. By induction on the number of points $p_i \geq 2$ of T. It is easy to observe that $P_{T_b}(T)$ of a path P with 2 points is $K_4 - x$, which is maximal outer planar. As the inductive hypothesis, let the pathos semitotal block graph of a non empty path P with n points be maximal outer planar. We now show that the pathos semitotal block graph of a path P' with $(n+1)$ points is maximal outer planar. First we prove that it is outer planar. Let the point and line sequence of the path

P' be $v_1, e_1, v_2, e_2, v_3, \ldots, v_n, e_n, v_{n+1}$, Where $v_1v_2 = e_1 = b_1, v_2v_3 = e_2 = b_2, \ldots, v_{n-1}v_n = e_{n-1} = b_{n1}, v_nv_{n+1} = e_n = b_n$.

The graphs P, P', $T_b(P)$, $T_b(P')$, $P_{T_b}(P)$ and $P_{T_b}(P')$ are shown in the figure 2. Without loss of generality $P' - v_{n+1} = P$.

By inductive hypothesis, $P_{T_b}(P)$ is maximal outer planar. Now the point v_{n+1} is one more point more in $P_{T_b}(P')$ than $P_{T_b}(P)$. Also there are only four lines $(v_{n+1}, v_n)(v_n, b_n)(b_n, v_{n+1})$ and (v_{n+1}, K_1) more in $P_{T_b}(P')$. Clearly the induced subgraph on the points v_{n+1}, v_n, b_n, K_1 is not K_4. Hence $P_{T_b}(P')$ is outer planar.

We now prove that $P_{T_b}(P')$ is maximal outer planar. Since $P_{T_b}(P)$ is maximal outer planar, it has $2(2q + k + 1) - 3$ lines. The outer planar graph $P_{T_b}(P')$ has $2(2q + k + 1) - 3 + 4 = 2(2q + k + 1 + 2) - 3$

$$= 2\left[(2q + 1) + (k + 1) + 1\right] - 3 \text{ lines.}$$

By Theorem [F], $P_{T_b}(P')$ is maximal outer planar. □

The next theorem gives a non-minimally non-outer planar $P_{T_b}(T)$.

Theorem 5 *For any non trivial tree T, the pathos semitotal block graph $P_{T_b}(T)$ of a tree T is non-minimally non-outer planar.*

Proof We have the following cases.

Case 1 Suppose T is a path, then $\Delta(T) \leq 2$, then by Theorem 4, $P_{T_B}(T)$ is maximal outer planar.

Case 2 Suppose T is not a path, then $\Delta(T) \geq 3$, then by theorem 3, $P_{T_b}(T)$ is planar. On embedding $P_{T_b}(T)$ in any plane, the points with degree greater than two of T forms the cut points. In $P_{T_b}(T)$ which lie on at least two blocks. Since each block of $P_{T_b}(T)$ is a maximal outer planar than one can easily verify that $P_{T_b}(T)$ is outer planar. Hence for any non trivial tree with $\Delta(T) \geq 3$, $P_{T_b}(T)$ is non minimally non-outer planar. □

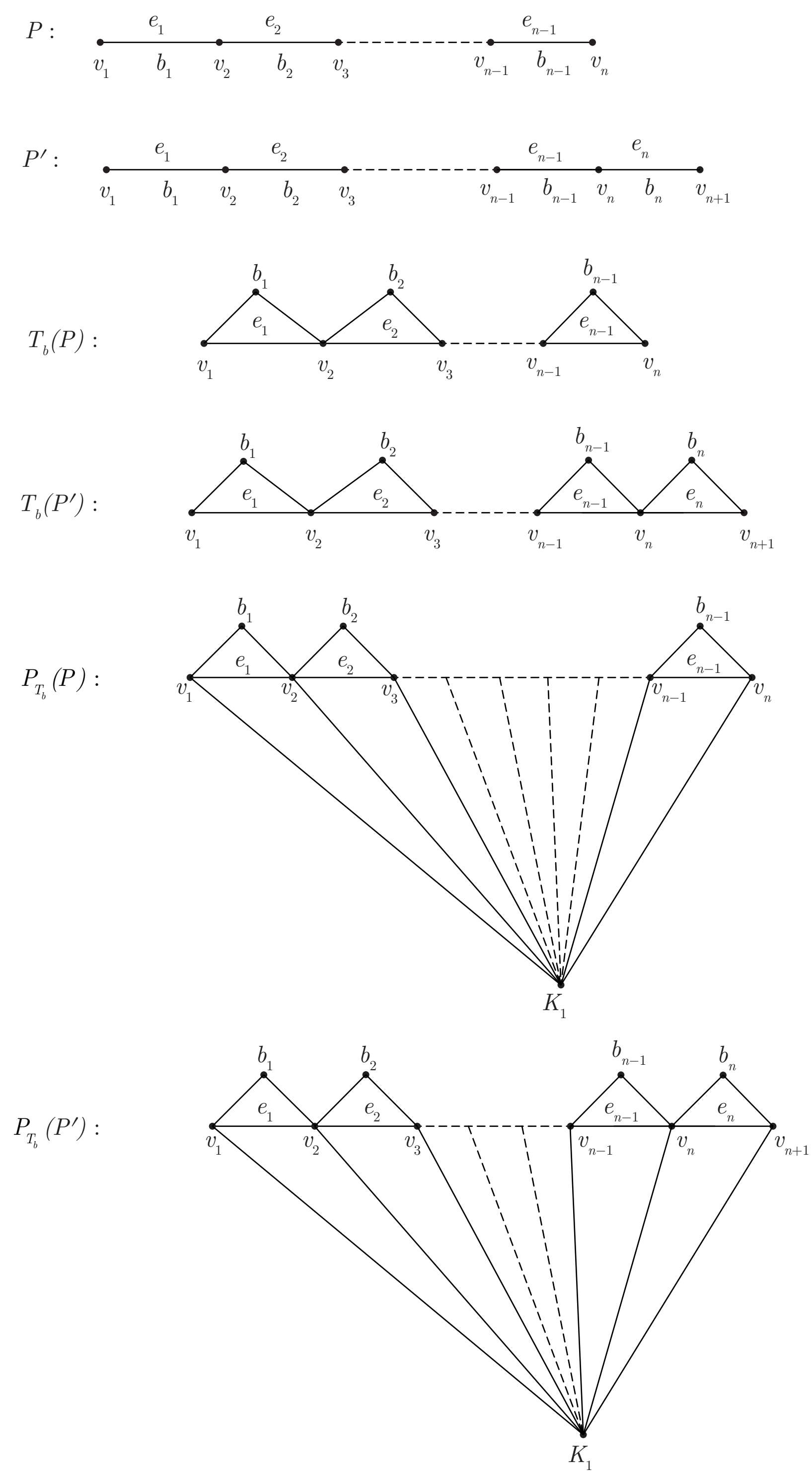
In the next theorem, we characterize the non-Eulerian $P_{T_b}(T)$.

Theorem 6 *For any non trivial tree T, the pathos semitotal block graph $P_{T_b}(T)$ of a tree T is non-Eulerian.*

Proof We have the following cases.

Case 1 Suppose T is a path with 2 points, then $P_{T_b}(T) = K_4 - x$, which is non-Eulerian. If T is a path with $p > 2$ points. Then in $T_b(T)$ each block is a triangle such that they are in sequence with the vertices of $T_b(T)$ as $\{v_1, b_1, v_2, v_1\}$ an induced subgraph as a triangle $T_b(T)$. Further $\{v_2, b_2, v_3, v_2\}$, $\{v_3, b_3, v_4, v_3\}, \ldots, \{v_{n-1}, b_n, v_n, v_{n-1}\}$, in which each set form a triangle as an induced subgraph of $T_b(T)$. Clearly one can easily verify that $T_b(T)$ is Eulerian. Now this path has exactly one pathos point say k_1, which is adjacent to $v_1, v_2, v_3, \ldots, v_n$ in $P_{T_b}(T)$ in which all the points $v_1, v_2, v_3, \ldots, v_n \in P_{T_b}(T)$ are of odd degree. Hence $P_{T_b}(T)$ is non-Eulerian.

Case 2 Suppose $\Delta(T) \geq 3$. Assume T has a unique point of degree ≥ 3 and also assume that $T = K_{1.n}$. Then in $T_b(T)$ each block is a triangle, such that the number of blocks which are K_3



P :
e_1
e_2
e_{n-1}
v_1
b_1
v_2
b_2
v_3
v_{n-1}
b_{n-1}
v_n
P' :
e_n
b_n
v_{n+1}
$T_b(P)$:
$T_b(P')$:
$P_{T_b}(P)$:
$P_{T_b}(P')$:
K_1

Figure 2:

are n with a common cut point k. Since the degree of a vertex $k = 2n$. One can easily verify that $T_b(K_{1,3})$ is Eulerian. To form $P_{T_b}(T)$, $T = K_{1,n}$, the points of degree 2 and the point k are joined by the corresponding pathos point which give $(n+1)$ points of odd degree in $P_{T_b}(T)$. Hence $P_{T_b}(T)$ is non-Eulerian. □

In the next theorem we characterize the hamiltonian $P_{T_b}(T)$.

Theorem 7 *For any non trivial tree T, the pathos semitotal block graph $P_{T_b}(T)$ of a tree T is hamiltonian if and only if T is a path.*

Proof For the necessity, suppose T is a path and has exactly one path of pathos. Let $V[T_b(T)] = \{u_1, u_2, u_3, \ldots, u_n\} \cup \{b_1, b_2, b_3, \ldots, b_{n-1}\}$, where $b_1, b_2, b_3, \ldots, b_{n-1}$ are block points of T. Since each block is a triangle and each block consists of points as $B_1 = \{u_1, b_1, u_2\}, B_2 = \{u_2, b_2, u_3\}, \ldots, B_m = \{u_m, b_m, u_{m+1}\}$. In $P_{T_b}(T)$ the pathos point w is adjacent to $\{u_1, u_2, u_3, \ldots, u_n\}$. Hence $V[P_{T_b}(T)] = \{u_1, u_2, u_3, \ldots, u_n\} \cup \{b_1, b_2, b_3, \ldots, b_{n-1}\} \cup w$ form a cycle as $w, u_1, b_1, u_2, b_2, u_2, \ldots u_{n-1}, b_{n-1}, u_n, w$. Containing all the points of $P_{T_b}(T)$. Clearly $P_{T_b}(T)$ is hamiltonian. Thus necessity is proved.

For the sufficiency, suppose $P_{T_b}(T)$ is hamiltonian, now we consider the following cases.

Case 1 Assume T is a path. Then T has at least one point with deg $v \geq 3$, $\forall v \in V(T)$, assume that T has exactly one point u such that degree $u > 2$, then $G = T = K_{1.n}$. Now we consider the following subcases of Case 1.

Subcase 1.1 For $K_{1.n}$, $n > 2$ and n is even, then in $T_b(T)$ each block is k_3. The number of path of pathos are $\frac{n}{2}$. Since n is even we get $\frac{n}{2}$ blocks. Each block contains two lines of $\langle K_4 - x \rangle$, which is a non line disjoint subgraph of $P_{T_b}(T)$. Since $P_{T_b}(T)$ has a cut point, one can easily verify that there does not exist any hamiltonian cycle, a contradiction.

Subcase 1.2 For $K_{1.n}$, $n > 2$ and n is odd, then the number of path of pathos are $\dfrac{n+1}{2}$, since n is odd we get $\dfrac{n-1}{2} + 1$ blocks in which $\dfrac{n-1}{2}$ blocks contains two times of $\langle K_4 - x \rangle$ which is nonline disjoint subgraph of $P_{T_b}(T)$ and remaining block is $\langle K_4 - x \rangle$. Since $P_{T_b}(T)$ contain a cut point, clearly $P_{T_b}(T)$ does not contain a hamiltonian cycle, a contradiction. Hence the sufficient condition.

§3. Pathos Total Block Graph of a Tree

A tree T, its total block graph $T_B(T)$, and their pathos total block graphs $P_{T_B}(T)$ are shown in the Fig.3. We start with a few preliminary results.

Remark 3 For any non trivial path, the inner point number of the pathos total block graph $P_{T_B}(T)$ of a tree T is equal to the number of cut points in T.

Remark 4 The degree of a pathos point in $P_{T_B}(T)$ is n, then the pathos length of the corresponding path P_i of pathos in T is $n-1$.

Remark 5 For any non trivial tree T, $P_{T_B}(T)$ is a block.

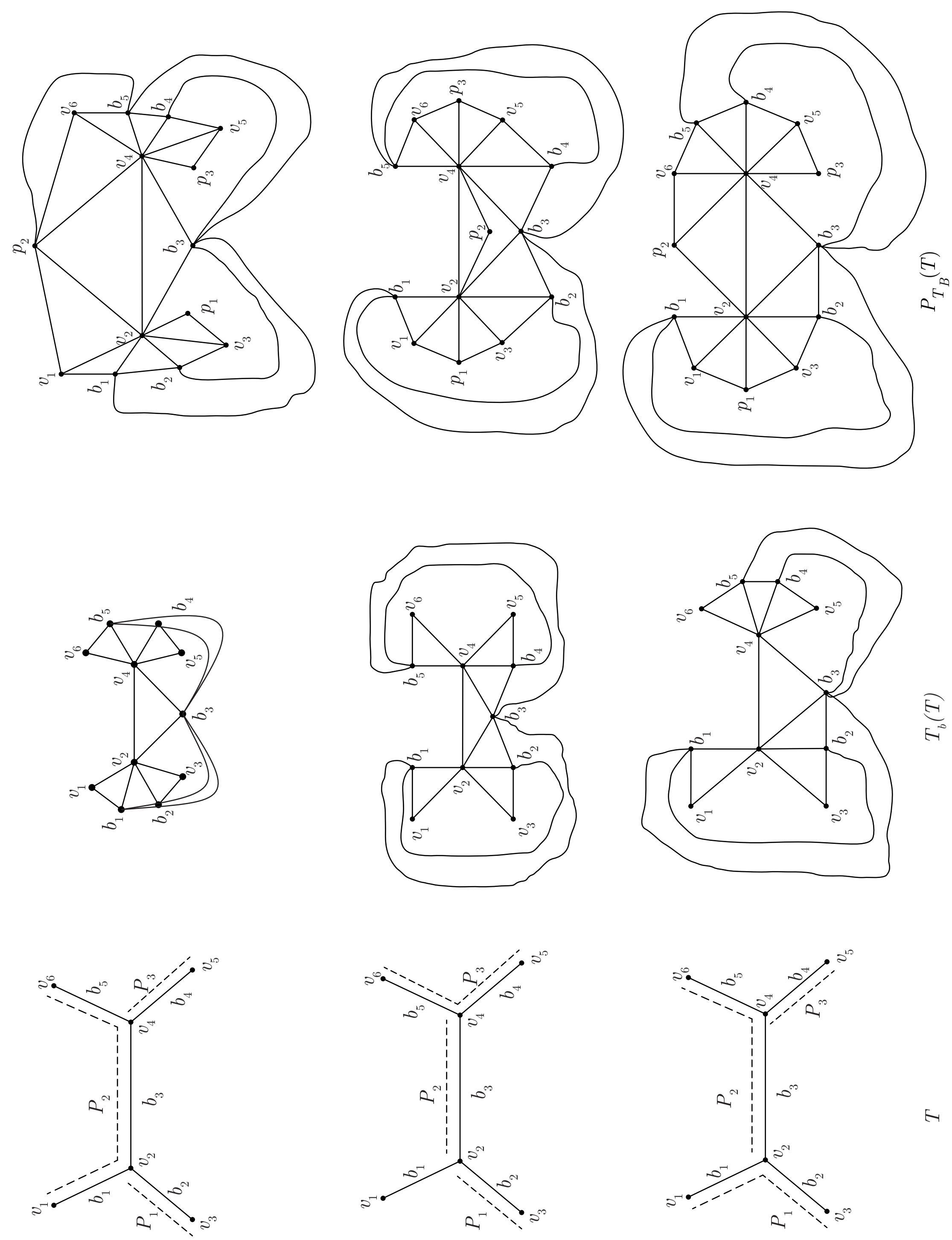


Figure 3:

Also in Kulli [4], developed the number of points and lines of a total block graph of a graph

has been expressed in terms of blocks of G. Now using this we have a modified theorem as shown below in which we have expressed the number of points and lines in terms of lines and degrees of the points of G which is a tree.

Theorem 8 *For any non trivial (p, q) tree whose points have degree d_i, the number of points and lines in total block graph of a tree T are $(2q+1)$ and $\left(2q + \frac{1}{2}\sum_{i=1}^{p} d_i^2\right)$.*

Proof By Theorem [B], the number of points in $T_b(T)$ is $\left(\sum_{i=1}^{b} b_i\right) + 1$, where b_i are the number of blocks in T to which the points v_i belongs in G. Since $\sum b_i = 2q$, for G is a tree. Thus the number of points in $T_B(G) = 2q+1$. Also, by Theorem [B], the number of lines in $T_B(G)$ are $q + \sum_{i=1}^{b} \binom{b_i+1}{2} = \left(\sum_{i=1}^{b} b_i\right) + \left(\frac{1}{2}\sum_{i=1}^{p} d_i^2\right) = \left(2q + \frac{1}{2}\sum_{i=1}^{p} d_i^2\right)$, for G is a tree. □

In the following theorem we obtain the number of points and lines in $P_{T_B}(T)$.

Theorem 9 *For any non trivial tree T, the pathos total block graph $P_{T_B}(T)$ of a tree T, whose points have degree d_i, then the number of points in $P_{T_B}(T)$ are $(2q+k+1)$ and the number of lines are $\left(q + 2 + \sum_{i=1}^{p} d_i^2\right)$, where k is the path number.*

Proof By Theorem 7, the number of points in $T_B(T)$ are $2q+1$, and by definition of $P_{T_B}(T)$, the number of points in $P_{T_B}(T)$ are $(2q+k+1)$, where k is the path number in T. Also by Theorem 7, the number of lines in $T_B(T)$ are $\left(2q + \frac{1}{2}\sum_{i=1}^{p} d_i^2\right)$. The number of lines in $P_{T_B}(T)$ is the sum of lines in $T_B(T)$ and the number of lines which lie on the points of pathos of T which are to $\left(-q + 2 + \frac{1}{2}\sum_{i=1}^{p} d_i^2\right)$. Thus the number of lines in $P_{T_B}(T)$ is equal to $\left(2q + \frac{1}{2}\sum_{i=1}^{p} d_i^2\right) + \left(-q + 2 + \frac{1}{2}\sum_{i=1}^{p} d_i^2\right) = \left(q + 2 + \sum_{i=1}^{p} d_i^2\right)$. □

§4. Planar Pathos Total Block Graphs

A criterion for pathos total block graph to be planar is presented in our next theorem.

Theorem 10 *For any non trivial tree T, the pathos total block graph $P_{T_B}(T)$ of a tree T is planar if and only if $\Delta(T) \leq 3$.*

Proof Suppose $P_{T_B}(T)$ is planar. Then by Theorem [C], each cut point of T lie on at most 3 blocks. Since each block is a line in a tree, now we can consider the degree of cutpoints instead of number of blocks incident with the cut points. Now suppose if $\Delta(T) \leq 3$, then by Theorem [C], $T_B(T)$ is planar. Let $\{b_1, b_2, b_3, \ldots, b_{p-1}\}$ be the blocks of T with p points such that $b_1 = e_1, b_2 = e_2, \ldots, b_{p-1} = e_{p-1}$ and P_i be the number of pathos of T. Now $V[P_{T_B}(T)] = V(G) \cup \{b_1, b_2, \ldots b_{p-1}\} \cup \{P_i\}$. By Theorem [C], and by the definition of pathos, the embedding of $P_{T_B}(T)$ in any plane gives a planar $P_{T_B}(T)$.

Suppose $\Delta(T) \geq 4$ and assume that $P_{T_B}(T)$ is planar. Then there exists at least one point

of degree 4, assume that there exists a vertex v such that $\deg v = 4$. Then in $T_B(T)$, this point together with the block points form k_5 as an induced subgraph. Further the corresponding pathos point are adjacent to the V(T) in $T_B(T)$ which gives $P_{T_B}(T)$ in which again k_5 as an induced subgraph, a contradiction to the planarity of $P_{T_B}(T)$. This completes the proof. □

The following theorem results the maximal outer planar $P_{T_B}(T)$.

Theorem 11 *For any non trivial tree T, the pathos total block graph $P_{T_B}(T)$ of a tree T is maximal outer planar if and only if $T = k_2$.*

Proof Suppose $T = k_3$ and $P_{T_B}(T)$ is maximal outer planar. Then $T_B(T) = k_4$ and one can easily verify that, $i[P_{T_B}(T)] > 1$, a contradiction. Further we assume that $T = K_{1,2}$ and $P_{T_B}(T)$ is maximal outer planar, then $T_B(T)$ is $W_p - x$, where x is outer line of W_p. Since $K_{1,2}$ has exactly one pathos, this point together with $W_p - x$ gives W_{p+1}. Clearly, $P_{T_B}(T)$ is non maximal outer planar, a contradiction. For the converse, if $T = k_2$, $T_B(T) = k_3$ and $P_{T_B}(T)$ $= K_4 - x$ which is a maximal outer planar. This completes the proof of the theorem. □

Now we have a pathos total block graph of a path $p \geq 2$ point as shown in the below remarks, and also a cycle with $p \geq 3$ points.

Remark 6 For any non trivial path with p points, $i[P_{T_B}(T)] = p - 2$.

Remark 7 For any cycle C_p, $p \geq 3$, $i[P_{T_B}(C_p)] = p - 1$.

The next theorem gives a minimally non-outer planar $P_{T_B}(T)$.

Theorem 12 *For any non trivial tree T, the pathos total block graph $P_{T_B}(T)$ of a tree T is minimally non-outer planar if and only if T is a path with 3 points.*

Proof Suppose $P_{T_B}(T)$ is minimally non-outer planar. Assume T is not a path. We consider the following cases.

Case 1 Suppose T is a tree with $\Delta(T) \geq 3$. Then there exists at least one point of degree at least 3. Assume v be a point of degree 3. Clearly, $T = K_{1,3}$. Then by the Theorem [D], $i[T_B(T)] > 1$ since $T_B(T)$ is a subgraph of $P_{T_B}(T)$. Clearly $i[P_{T_B}(T)] \geqslant 2$ a contradiction.

Case 2 Suppose T is a closed path with p points, then it is a cycle with p points. By Theorem [D], $P_{T_B}(T)$ is minimally non-outer planar. By Remark 7, $i[P_{T_B}(T)] > 1$, a contradiction.

Case 3 Suppose T is a closed path with $p \geq 4$ points, clearly by Remark 6, $i[P_{T_B}(T)] > 2$, a contradiction.

Conversely, suppose T is a path with 3 points, clearly by Remark 6, $i[P_{T_B}(T)] = 1$. This gives the required result. □

In the following theorem we characterize the non-Eulerian $P_{T_B}(T)$.

Theorem 13 *For any non trivial tree T, the pathos total block graph $P_{T_B}(T)$ of a tree T is non-Eulerian.*

Proof We consider the following cases.

Case 1 Suppose T is a path. For $p = 2$ points, then $P_{T_B}(T) = K_4 - x$, which is non-Eulerian. For $p = 3$ points, then $P_{T_B}(T)$ is a wheel, which is non-Eulerian.

For $p \geq 4$ we have a path $P : v_1, v_2, v_3, \ldots, v_p$. Now in path each line is a block. Then $v_1v_2 = e_1 = b_1, v_2v_3 = e_2 = b_2, \ldots, v_{p-1}v_p = e_{p-1} = b_{p-1}$, $\forall e_{p-1} \in E(G)$, and $\forall b_{p-1} \in V[T_B(P)]$. In $T_B(P)$, the degree of each point is even except b_1 and b_{p-1}. Since the path P has exactly one pathos which is a point of $P_{T_B}(P)$ and is adjacent to the points $v_1, v_2, v_3, \ldots, v_p$, of $T_B(P)$ which are of even degree, gives as an odd degree points in $P_{T_B}(P)$ including odd degree points b_1 and b_2. Clearly $P_{T_B}(P)$ is non-Eulerian.

Case 2 Suppose T is not a path. We consider the following subcases,

Subcase 2.1 Assume T has a unique point degree ≥ 3 and $T = K_{1.n}$, where n is odd. Then in $T_B(T)$ each block is a triangle such that there are n number of triangles with a common cut point k which has a degree $2n$. Since the degree of each point in $T_B(K_{1,n})$ is Eulerian. To form $P_{T_B}(T)$ where $T = K_{1,n}$, the points of degree 2 and the point k are joined by the corresponding pathos point which gives $(n+1)$ points of odd degree in $P_{T_B}(K_{1.n})$. $P_{T_B}(K_{1.n})$ has n points of odd degree. Hence $P_{T_B}(T)$ non-Eulerian.

Assume that $T = K_{1.n}$, where n is even, then in $T_B(T)$ each block is a triangle, which are $2n$ in number with a common cut point k. Since the degree of each point other than k is either 2 or $(n+1)$ and the degree of the point k is $2n$. One can easily verify that $T_B(K_{1,n})$ is non-Eulerian. To form $P_{T_B}(T)$ where $T = K_{1,n}$, the points of degree 2 and the point k are joined by the corresponding pathos point which gives $(n+2)$ points of odd degree in $P_{T_B}(T)$. Hence $P_{T_B}(T)$ non-Eulerian.

Subcase 2.2 Assume T has at least two points of degree ≥ 3. Then $V[T_B(T)] = V(G) \cup \{b_1, b_2, b_3, \ldots, b_p\}$, $\forall e_p \in E(G)$. In $T_B(T)$, each endpoint has degree 2 and these points are adjacent to the corresponding pathos points in $P_{T_B}(T)$ gives degree 3, From Case 1, Tree T has at least 4 points and by Corollary [A], $P_{T_B}(T)$ has at least two points of degree 3. Hence $P_{T_B}(T)$ is non-Eulerian. □

In the next theorem we characterize the hamiltonian $P_{T_B}(T)$.

Theorem 14 *For any non trivial tree T, the pathos total block graph $P_{T_B}(T)$ of a tree T is hamiltonian.*

Proof We consider the following cases.

Case 1 Suppose T is a path with $\{u_1, u_2, u_3, \ldots, u_n\} \in V(T)$ and $b_1, b_2, b_3, \ldots, b_m$ be the number of blocks of T such that $m = n - 1$. Then it has exactly one path of pathos. Now point set of $T_B(T)$ is $V[T_B(T)] = \{u_1, u_2, \ldots, u_n\} \cup \{b_1, b_2, \ldots, b_m\}$. Since given graph is a path then in $T_B(T)$, $b_1 = e_1, b_2 = e_2, \ldots, b_m = e_m$, such that $b_1, b_2, b_3, \ldots, b_m \subset V[T_B(T)]$. Then by the definition of total block graph $\{u_1, u_2, \ldots, u_m\} \cup \{b_1, b_2, \ldots, b_{m-1}, b_m\} \cup \{b_1, u_1, b_2u_2, \ldots, b_mu_{n-1}, b_mu_n\}$ form line set of $T_B(T)$[see Fig. 4].

Now this path has exactly one pathos say w. In forming pathos total block graph of a path, the pathos w becomes a point, then $V[P_{T_B}(T)] = \{u_1, u_2, \ldots, u_n\} \cup \{b_1, b_2, \ldots, b_m\} \cup \{w\}$ and

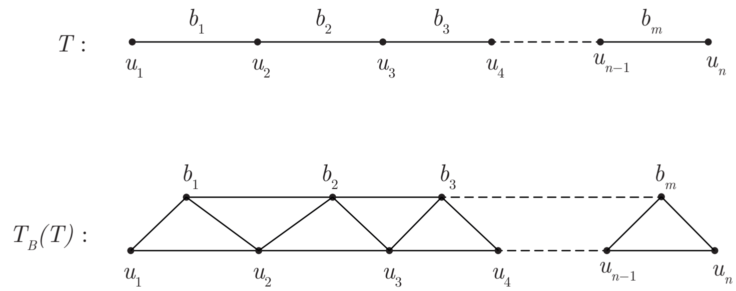


Figure 4:

w is adjacent to all the points $\{u_1, u_2, \ldots, u_m\}$ shown in the Fig.5.

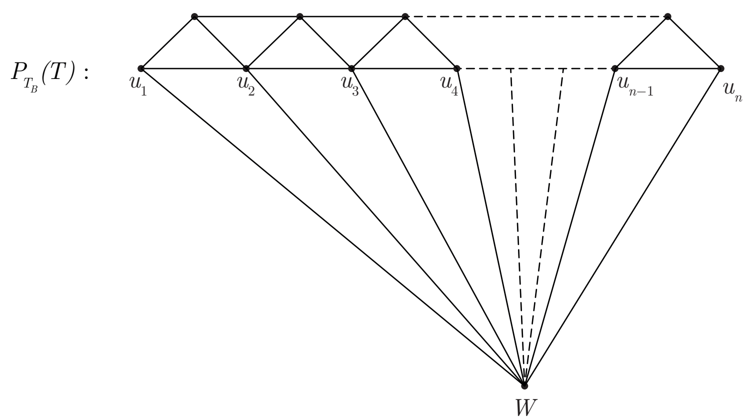


Figure 5:

In $P_{T_B}(T)$, the hamiltonian cycle $w, u_1, b_1, u_2, b_2, u_2, u_3, b_3, \ldots, u_{n-1}, b_m, u_n, w$ exist. Clearly the pathos total block graph of a path is hamiltonian graph.

Case 2 Suppose T is not a path. Then T has at least one point with degree at least 3. Assume that T has exactly one point u such that degree> 2. Now we consider the following subcases of case 2.

Subcase 2.1 Assume $T = K_{1.n}$, $n > 2$ and is odd. Then the number of paths of pathos are $\frac{n+1}{2}$. Let $V\left[T_B\left(T\right)\right] = \{u_1, u_2, \ldots, u_n, b_1, b_2, \ldots, b_{m-1}\}$. By the definition of $P_{T_B}(T)$, $V[P_{T_B}(T)] = \{u_1, u_2, \ldots, u_n, b_1 b_2, \ldots, b_{n-1}\} \cup \{p_1, p_2, \ldots, p_{n+1/2}\}$. Then there exists a cycle containing the points of $P_{T_B}(T)$ as $p_1, u_1, b_1, b_2, u_3, p_2, u_2, b_3, u_4, \ldots p_1$ and is a hamiltonian cycle. Hence $P_{T_B}(T)$ is a hamiltonian.

Subcase 2.2 Assume $T = K_{1.n}$, $n > 2$ and is even. Then the number of path of pathos are $\frac{n}{2}$, then $V\left[T_B\left(T\right)\right] = \{u_1, u_2, \ldots, u_n, b_1, b_2, \ldots b_{n-1}\}$. By the definition of $P_{T_B}(T)$. $V[P_{T_B}(T)] = \{u_1, u_2, \ldots, u_n, b_1, b_2, \ldots, b_{n-1}\} \cup \{p_1, p_2, \ldots, p_{n/2}\}$. Then there exist a cycle containing the points of $P_{T_B}(T)$ as $p_1, u_1, b_1, b_2, u_3, p_2, u_4, b_3, b_4, \ldots, p_1$ and is a hamiltonian cycle. Hence $P_{T_B}(T)$ is a hamiltonian.

Suppose T is neither a path or a star. Then T contains at least two points of degree> 2. Let $u_1, u_2, u_3, \ldots, u_n$ be the points of degree ≥ 2 and $v_1, v_2, v_3, \ldots, v_m$ be the end points of T. Since end block is a line in T, and denoted as $b_1, b_2, \ldots, b_k$, then tree T has p_i pathos points, $i > 1$ and each pathos point is adjacent to the point of T where the corresponding pathos lie on the points of T. Let $\{p_1, p_2, \ldots.., p_i\}$ be the pathos points of T. Then there exists a cycle C containing all the points of $P_{T_B}(T)$ as $p_1, v_1, b_1, b_2, v_2, p_2, u_1, b_3, u_2, p_3, v_3, b_4, \ldots, v_{n-1}, b_{n-1}, b_n, v_n, \ldots, p_1$. Hence $P_{T_B}(T)$ is a hamiltonian cycle. Hence $P_{T_B}(T)$ is a hamiltonian graph. □

In the next theorem we characterize $P_{T_B}(T)$ in terms of crossing number one.

Theorem 15 *For any non trivial tree T, the pathos total block graph $P_{T_B}(T)$ of a tree T has crossing number one if and only if $\Delta(T) \leq 4$, and there exist a unique point in T of degree 4.*

Proof Suppose $P_{T_B}(T)$ has crossing number one. Then it is non-planar. Then by Theorem 10, we have $\Delta(T) \geq 4$. We now consider the following cases.

Case 1 Assume $\Delta(T) = 5$. Then by Theorem [E], $T_B\,(T)$ is non-planar with crossing number more than one. Since $T_B\,(T)$ is a subgraph of $P_{T_B}(T)$. Clearly $\mathrm{cr}(P_{T_B}\,(T)) > 1$, a contradiction.

Case 2 Assume $\Delta(T) = 4$. Suppose T has two points of degree 4. Then by Theorem [E], $T_B\,(T)$ has crossing number at least two. But $T_B\,(T)$ is a subgraph of $P_{T_B}(T)$. Hence $\mathrm{cr}(P_{T_B}\,(T)) > 1$, a contradiction.

Conversely, suppose T satisfies the given condition and assume T has a unique point v of degree 4. The lines which are blocks in T such that they are the points in $T_B\,(T)$. In $T_B\,(T)$, these block points and a point v together forms an induced subgraph as k_5. In forming $P_{T_B}(T)$, the pathos points are adjacent to at most two points of this induced subgraph. Hence in all these cases the $\mathrm{cr}(P_{T_B}\,(T)) = 1$. This completes the proof. □

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Varieties of Groupoids and Quasigroups Generated by Linear-Bivariate Polynomials Over Ring $\mathbb{Z}_n$

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Abstract: Some varieties of groupoids and quasigroups generated by linear-bivariate polynomials $P(x, y) = a+bx+cy$ over the ring $\mathbb{Z}_n$ are studied. Necessary and sufficient conditions for such groupoids and quasigroups to obey identities which involve one, two, three (e.g. Bol-Moufang type) and four variables w.r.t. a, b and c are established. Necessary and sufficient conditions for such groupoids and quasigroups to obey some inverse properties w.r.t. a, b and c are also established. This class of groupoids and quasigroups are found to belong to some varieties of groupoids and quasigroups such as medial groupoid(quasigroup), F-quasigroup, semi automorphic inverse property groupoid(quasigroup) and automorphic inverse property groupoid(quasigroup).

Key Words: groupoids, quasigroups, linear-bivariate polynomials.

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§1. Introduction

1.1 Groupoids, Quasigroups and Identities

Let G be a non-empty set. Define a binary operation $(\cdot)$ on G. $(G, \cdot)$ is called a groupoid if G is closed under the binary operation $(\cdot)$. A groupoid $(G, \cdot)$ is called a quasigroup if the equations $a \cdot x = b$ and $y \cdot c = d$ have unique solutions for x and y for all $a, b, c, d \in G$. A quasigroup $(G, \cdot)$ is called a loop if there exists a unique element $e \in G$ called the identity element such that $x \cdot e = e \cdot x = x$ for all $x \in G$.

A function $f : S \times S \to S$ on a finite set S of size $n > 0$ is said to be a Latin square (of order n) if for any value $a \in S$ both functions $f(a, \cdot)$ and $f(\cdot, a)$ are permutations of S. That is, a Latin square is a square matrix with n^2 entries of n different elements, none of them occurring more than once within any row or column of the matrix.

Definition 1.1 *A pair of Latin squares $f_1(\cdot, \cdot)$ and $f_2(\cdot, \cdot)$ is said to be orthogonal if the pairs $\big(f_1(x, y), f_2(x, y)\big)$ are all distinct, as x and y vary.*

For associative binary systems, the concept of an inverse element is only meaningful if the

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system has an identity element. For example, in a group $(G,\cdot)$ with identity element $e\in G$, if $x\in G$ then the inverse element for x is the element $x^{-1}\in G$ such that

$$x\cdot x^{-1}=x^{-1}\cdot x=e.$$

In a loop $(G,\cdot)$ with identity element e, the left inverse element of $x\in G$ is the element $x^{\lambda}\in G$ such that

$$x^{\lambda}\cdot x=e$$

while the right inverse element of $x\in G$ is the element $x^{\rho}\in G$ such that

$$x\cdot x^{\rho}=e$$

In case $(G,\cdot)$ is a quasigroup, then for each $x\in G$, the elements $x^{\rho}\in G$ and $x^{\lambda}\in G$ such that $xx^{\rho}=e^{\rho}$ and $x^{\lambda}x=e^{\lambda}$ are called the right and left inverse elements of x respectively. Here, $e^{\rho}\in G$ and $e^{\lambda}\in G$ satisfy the relations $xe^{\rho}=x$ and $e^{\lambda}x=x$ for all $x\in G$ and are respectively called the right and left identity elements. Whenever $e^{\rho}=e^{\lambda}$, then $(G,\cdot)$ becomes a loop.

In case $(G,\cdot)$ is a groupoid, then for each $x\in G$, the elements $x^{\rho}\in G$ and $x^{\lambda}\in G$ such that $xx^{\rho}=e_{\rho}(x)$ and $x^{\lambda}x=e_{\lambda}(x)$ are called the right and left inverse elements of x respectively. Here, $e_{\rho}(x)\in G$ and $e_{\lambda}(x)\in G$ satisfy the relations $xe_{\rho}(x)=x$ and $e_{\lambda}(x)x=x$ for each $x\in G$ and are respectively called the local right and local left identity elements of x. Whenever $e_{\rho}(x)=e_{\lambda}(x)$, then we simply write $e(x)=e_{\rho}(x)=e_{\lambda}(x)$ and call it the local identity of x.

The basic text books on quasigroups, loops are Pflugfelder [19], Bruck [1], Chein, Pflugfelder and Smith [2], Dene and Keedwell [3], Goodaire, Jespers and Milies [4], Sabinin [25], Smith [26], Jaíyéọlá [5] and Vasantha Kandasamy [28].

Groupoids, quasigroups and loops are usually studied relative to properties or identities. If a groupoid, quasigroup or loop obeys a particular identity, then such types of groupoids, quasigroups or loops are said to form a variety. In this work, our focus will be on groupoids and quasigroups. Some identities that describe groupoids and quasigroups which would be of interest to us here are categorized as follows:

(A) Those identities which involve one element only on each side of the equality sign:

$$aa=a \qquad \text{idempotent law} \tag{1}$$

$$aa=bb \qquad \text{unipotent law} \tag{2}$$

(B) Those identities which involve two elements on one or both sides of the equality sign:

$$ab=ba \qquad \text{commutative law} \tag{3}$$

$$(ab)b=a \qquad \text{Sade right Keys law} \tag{4}$$

$$b(ba)=a \qquad \text{Sade left keys law} \tag{5}$$

$$(ab)b=a(bb) \qquad \text{right alternative law} \tag{6}$$

$$b(ba)=(bb)a \qquad \text{left alternative law} \tag{7}$$

$a(ba) = (ab)a$	medial alternative law	(8)
$a(ba) = b$	law of right semisymmetry	(9)
$(ab)a = b$	law of left semisymmetry	(10)
$a(ab) = ba$	Stein first law	(11)
$a(ba) = (ba)a$	Stein second law	(12)
$a(ab) = (ab)b$	Schroder first law	(13)
$(ab)(ba) = a$	Schroder second law	(14)
$(ab)(ba) = b$	Stein third law	(15)
$ab = a$	Sade right translation law	(16)
$ab = b$	Sade left translation law	(17)

(C) Those identities which involve three distinct elements on one or both sides of the equality sign:

$(ab)c = a(bc)$	associative law	(18)
$a(bc) = c(ab)$	law of cyclic associativity	(19)
$(ab)c = (ac)b$	law of right permutability	(20)
$a(bc) = b(ac)$	law of left permutability	(21)
$a(bc) = c(ba)$	Abel-Grassman law	(22)
$(ab)c = a(cb)$	commuting product law	(23)
$c(ba) = (bc)a$	dual of commuting product	(24)
$(ab)(bc) = ac$	Stein fourth law	(25)
$(ba)(ca) = bc$	law of right transitivity	(26)
$(ab)(ac) = bc$	law of left transitivity	(27)
$(ab)(ac) = cb$	Schweitzer law	(28)
$(ba)(ca) = cb$	dual of Schweitzer law	(29)
$(ab)c = (ac)(bc)$	law of right self-distributivity law	(30)
$c(ba) = (cb)(ca)$	law of left self-distributivity law	(31)
$(ab)c = (ca)(bc)$	law of right abelian distributivity	(32)
$c(ba) = (cb)(ac)$	law of left abelian distributivity	(33)
$(ab)(ca) = [a(bc)]a$	Bruck-Moufang identity	(34)
$(ab)(ca) = a[bc)a]$	dual of Bruck-Moufang identity	(35)
$[(ab)c]b = a[b(cb)]$	Moufang identity	(36)

$$[(bc)b]a = b[c(ba)] \quad \text{Moufang identity} \tag{37}$$

$$[(ab)c]b = a[(bc)b] \quad \text{right Bol identity} \tag{38}$$

$$[b(cb)]a = b[c(ba)] \quad \text{left Bol identity} \tag{39}$$

$$[(ab)c]a = a[b(ca)] \quad \text{extra law} \tag{40}$$

$$[(ba)a]c = b[(aa)c] \quad \text{RC}_4 \text{ law} \tag{41}$$

$$[b(aa)]c = b[a(ac)] \quad \text{LC}_4 \text{ law} \tag{42}$$

$$(aa)(bc) = [a(ab)]c \quad \text{LC}_2 \text{ law} \tag{43}$$

$$[(bc)a]a = b[(ca)a] \quad \text{RC}_1 \text{ law} \tag{44}$$

$$[a(ab)]c = a[a(bc)] \quad \text{LC}_1 \text{ law} \tag{45}$$

$$(bc)(aa) = b[(ca)a] \quad \text{RC}_2 \text{ law} \tag{46}$$

$$[(aa)b]c = a[a(bc)] \quad \text{LC}_3 \text{ law} \tag{47}$$

$$[(bc)a]a = b[c(aa)] \quad \text{RC}_3 \text{ law} \tag{48}$$

$$[(ba)a]c = b[a(ac)] \quad \text{C-law} \tag{49}$$

$$a[b(ca)] = cb \quad \text{Tarski law} \tag{50}$$

$$a[(bc)(ba)] = c \quad \text{Neumann law} \tag{51}$$

$$(ab)(ca) = (ac)(ba) \quad \text{specialized medial law} \tag{52}$$

(D) Those involving four elements:

$$(ab)(cd) = (ad)(cb) \quad \text{first rectangle rule} \tag{53}$$

$$(ab)(ac) = (db)(dc) \quad \text{second rectangle rule} \tag{54}$$

$$(ab)(cd) = (ac)(bd) \quad \text{internal mediality or medial law} \tag{55}$$

(E) Those involving left or right inverse elements:

$$x^\lambda \cdot xy = y \quad \text{left inverse property} \tag{56}$$

$$yx \cdot x^\rho = y \quad \text{right inverse property} \tag{57}$$

$$x(yx)^\rho = y^\rho \text{ or } (xy)^\lambda x = y^\lambda \quad \text{weak inverse property(WIP)} \tag{58}$$

$$xy \cdot x^\rho = y \text{ or } x \cdot yx^\rho = y \text{ or } x^\lambda \cdot (yx) = y \text{ or } x^\lambda y \cdot x = y \text{ cross inverse property(CIP)} \tag{59}$$

$$(xy)^\rho = x^\rho y^\rho \text{ or } (xy)^\lambda = x^\lambda y^\lambda \text{ automorphic inverse property (AIP)} \tag{60}$$

$$(xy)^\rho = y^\rho x^\rho \text{ or } (xy)^\lambda = y^\lambda x^\lambda \text{ anti-automorphic inverse property (AAIP)} \tag{61}$$

$$(xy \cdot x)^\rho = x^\rho y^\rho \cdot x^\rho \text{ or } (xy \cdot x)^\lambda = x^\lambda y^\lambda \cdot x^\lambda \text{ semi-automorphic inverse property (SAIP)} \tag{62}$$

Definition 1.2(Trimedial Quasigroup) *A quasigroup is trimedial if every subquasigroup generated by three elements is medial.*

Medial quasigroups have also been called abelian, entropic, and other names, while trimedial quasigroups have also been called triabelian, terentropic, etc.

There are two distinct, but related, generalizations of trimedial quasigroups. The variety of semimedial quasigroups(also known as weakly abelian, weakly medial, etc.) is defined by the equations

$$xx \cdot yz = xy \cdot xz \tag{63}$$

$$zy \cdot xx = zx \cdot yz \tag{64}$$

Definition 1.3(Semimedial Quasigroup) *A quasigroup satisfying (63) (resp. (64) is said to be left (resp. right) semimedial.*

Definition 1.4(Medial-Like Identities) *A groupoid or quasigroup is called an external medial groupoid or quasigroup if it obeys the identity*

$$ab \cdot cd = db \cdot ca \qquad \textit{external medial or paramediality law} \tag{65}$$

A groupoid or quasigroup is called a palindromic groupoid or quasigroup if it obeys the identity

$$ab \cdot cd = dc \cdot ba \qquad \textit{palidromity law} \tag{66}$$

Other medial like identities of the form $(ab)(cd) = (\pi(a)\pi(b))(\pi(c)\pi(d))$, where π is a certain permutation on $\{a, b, c, d\}$ are given as follows:

$$ab \cdot cd = ab \cdot dc \qquad C_1 \tag{67}$$

$$ab \cdot cd = ba \cdot cd \qquad C_2 \tag{68}$$

$$ab \cdot cd = ba \cdot dc \qquad C_3 \tag{69}$$

$$ab \cdot cd = cd \cdot ab \qquad C_4 \tag{70}$$

$$ab \cdot cd = cd \cdot ba \qquad C_5 \tag{71}$$

$$ab \cdot cd = dc \cdot ab \qquad C_6 \tag{72}$$

$$ab \cdot cd = ac \cdot db \qquad CM_1 \tag{73}$$

$$ab \cdot cd = ad \cdot bc \qquad CM_2 \tag{74}$$

$$ab \cdot cd = ad \cdot cb \qquad CM_3 \tag{75}$$

$$ab \cdot cd = bc \cdot ad \qquad CM_4 \tag{76}$$

$$ab \cdot cd = bc \cdot da \qquad CM_5 \tag{77}$$

$$ab \cdot cd = bd \cdot ac \qquad CM_6 \tag{78}$$

$$ab \cdot cd = bd \cdot ca \qquad CM_7 \tag{79}$$

$$ab \cdot cd = ca \cdot bd \qquad CM_8 \tag{80}$$

$$ab \cdot cd = ca \cdot db \qquad CM_9 \tag{81}$$

$$ab \cdot cd = cb \cdot ad \qquad CM_{10} \tag{82}$$

$$ab \cdot cd = cb \cdot da \qquad CM_{11} \tag{83}$$

$$ab \cdot cd = da \cdot bc \qquad CM_{12} \tag{84}$$

$$ab \cdot cd = da \cdot cb \qquad CM_{13} \tag{85}$$

$$ab \cdot cd = db \cdot ac \qquad CM_{14} \tag{86}$$

The variety of F-quasigroups was introduced by Murdoch [18].

Definition 1.5(F-quasigroup) *An F-quasigroup is a quasigroup that obeys the identities*

$$x \cdot yz = xy \cdot (x\backslash x)z \qquad \textit{left F-law} \tag{87}$$

$$zy \cdot x = z(x/x) \cdot yx \qquad \textit{right F-law} \tag{88}$$

A quasigroup satisfying (87) (resp. (77)) is called a left (resp. right) F-quasigroup.

Definition 1.6(E-quasigroup) *An E-quasigroup is a quasigroup that obeys the identities*

$$x \cdot yz = e_\lambda(x)y \cdot xz \qquad E_l \textit{ law} \tag{89}$$

$$zy \cdot x = zx \cdot ye_\rho(x) \qquad E_r \textit{ Law} \tag{90}$$

A quasigroup satisfying (89) (resp. (90)) is called a left (resp. right) E-quasigroup.

Some identities will make a quasigroup to be a loop, such are discussed in Keedwell [6]-[7].

Definition 1.7(Linear Quasigroup and T-quasigroup) *A quasigroup $(Q, \cdot)$ of the form $x \cdot y = x\alpha + y\beta + c$ where $(Q, +)$ is a group, α is its automorphism and β is a permutation of the set Q, is called a left linear quasigroup.*

A quasigroup $(Q, \cdot)$ of the form $x \cdot y = x\alpha + y\beta + c$ where $(Q, +)$ is a group, β is its automorphism and α is a permutation of the set Q, is called a right linear quasigroup.

A T-quasigroup is a quasigroup $(Q, \cdot)$ defined over an abelian group $(Q, +)$ by $x \cdot y = c + x\alpha + y\beta$, where c is a fixed element of Q and α and β are both automorphisms of the group $(Q, +)$.

Whenever one considers mathematical objects defined in some abstract manner, it is usually desirable to determine that such objects exist. Although occasionally this is accomplished by means of an abstract existential argument, most frequently, it is carried out through the presentation of a suitable example, often one which has been specifically constructed for the purpose. An example is the solution to the open problem of the axiomization of rectangular quasigroups and loops by Kinyon and Phillips [12] and the axiomization of trimedial quasigroups by Kinyon and Phillips [10], [11].

Chein et. al. [2] presents a survey of various methods of construction which has been used in the literature to generate examples of groupoids and quasigroups. Many of these constructions are ad hoc-designed specifically to produce a particular example; while others are of more general applicability. More can be found on the construction of (r, s, t)-inverse quasigroups in Keedwell and Shcherbacov [8]-[9], idempotent medial quasigroups in Krčadinac and Volenec [14] and quasigroups of Bol-Moufang type in Kunen [15]-[16].

Remark 1.1 In the survey of methods of construction of varieties and types of quasigroups highlighted in Chein et. al. [2], it will be observed that some other important types of quasigroups that obey identities (1) to (90) are not mentioned. Also, examples of methods of construction of such varieties that are groupoids are also scarce or probably not in existence by our search. In Theorem 1.4 of Kirnasovsky [13], the author characterized T-quasigroups with a score and two identities from among identities (1) to (90). The present work thus proves some results with which such groupoids and quasigroups can be constructed.

1.2 Univariate and Bivariate Polynomials

Consider the following definitions.

Definition 1.8 *A polynomial* $P(x) = a_0 + a_1x + \cdots + a_nx^n$, $n \in \mathbb{N}$ *is said to be a permutation polynomial over a finite ring* R *if the mapping defined by* P *is a bijection on* R.

Definition 1.9 *A bivariate polynomial is a polynomial in two variables,* x *and* y *of the form* $P(x, y) = \Sigma_{i,j}a_{ij}x^iy^j$.

Definition 1.10(Bivariate Polynomial Representing a Latin Square) *A bivariate polynomial* $P(x, y)$ *over* $\mathbb{Z}_n$ *is said to represent (or generate) a Latin square if* $(\mathbb{Z}_n, *)$ *is a quasigroup where* $* : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n$ *is defined by* $x * y = P(x, y)$ *for all* $x, y \in \mathbb{Z}_n$.

Mollin and Small [17] considered the problem of characterizing permutation polynomials. They established conditions on the coefficients of a polynomial which are necessary and sufficient for it to represent a permutation.

Shortly after, Rudolf and Mullen [23] provided a brief survey of the main known classes of permutation polynomials over a finite field and discussed some problems concerning permutation polynomials (PPs). They described several applications of permutations which indicated why the study of permutations is of interest. Permutations of finite fields have become of considerable interest in the construction of cryptographic systems for the secure transmission of data. Thereafter, the same authors in their paper [24], described some results that had appeared after their earlier work including two major breakthroughs.

Rivest [22] studied permutation polynomials over the ring $(\mathbb{Z}_n, +, \cdot)$ where n is a power of 2: $n = 2^w$. This is based on the fact that modern computers perform computations modulo 2^w efficiently (where $w = 2, 8, 16, 32$ or 64 is the word size of the machine), and so it was of interest to study PPs modulo a power of 2. Below is an important result from his work which is relevant to the present study.

Theorem 1.1(Rivest [22]) *A bivariate polynomial $P(x,y) = \Sigma_{i,j} a_{ij} x^i y^j$ represents a Latin square modulo $n = 2^w$, where $w \geq 2$, if and only if the four univariate polynomials $P(x,0)$, $P(x,1)$, $P(0,y)$, and $P(1,y)$ are all permutation polynomial modulo n.*

Vadiraja and Shankar [27] motivated by the work of Rivest continued the study of permutation polynomials over the ring $(\mathbb{Z}_n, +, \cdot)$ by studying Latin squares represented by linear and quadratic bivariate polynomials over $\mathbb{Z}_n$ when $n \neq 2^w$ with the characterization of some PPs. Some of the main results they got are stated below.

Theorem 1.2(Vadiraja and Shankar [27]) *A bivariate linear polynomial $a + bx + cy$ represents a Latin square over $\mathbb{Z}_n$, $n \neq 2^w$ if and only if one of the following equivalent conditions is satisfied:*

(i) *both b and c are coprime with n;*

(ii) *$a + bx$, $a + cy$, $(a + c) + bx$ and $(a + b) + cy$ are all permutation polynomials modulo n.*

Remark 1.2 It must be noted that $P(x,y) = a+bx+cy$ represents a groupoid over $\mathbb{Z}_n$. $P(x,y)$ represents a quasigroup over $\mathbb{Z}_n$ if and only if $(\mathbb{Z}_n, P)$ is a T-quasigroup. Hence whenever $(\mathbb{Z}_n, P)$ is a groupoid and not a quasigroup, $(\mathbb{Z}_n, P)$ is neither a T-quasigroup nor left linear quasigroup nor right linear quasigroup. Thus, the present study considers both T-quasigroup and non-T-quasigroup.

Theorem 1.3(Vadiraja and Shankar [27]) *If $P(x,y)$ is a bivariate polynomial having no cross term, then $P(x,y)$ gives a Latin square if and only if $P(x,0)$ and $P(0,y)$ are permutation polynomials.*

The authors were able to establish the fact that Rivest's result for a bivariate polynomial over $\mathbb{Z}_n$ when $n = 2^w$ is true for a linear-bivariate polynomial over $\mathbb{Z}_n$ when $n \neq 2^w$. Although the result of Rivest was found not to be true for quadratic-bivariate polynomials over $\mathbb{Z}_n$ when $n \neq 2^w$ with the help of counter examples, nevertheless some of such squares can be forced to be Latin squares by deleting some equal numbers of rows and columns.

Furthermore, Vadiraja and Shankar [27] were able to find examples of pairs of orthogonal Latin squares generated by bivariate polynomials over $\mathbb{Z}_n$ when $n \neq 2^w$ which was found impossible by Rivest for bivariate polynomials over $\mathbb{Z}_n$ when $n = 2^w$.

1.4 Some Important Results on Medial-Like Identities

Some important results which we would find useful in our study are stated below.

Theorem 1.4(Polonijo [21]) *For any groupoid $(Q, \cdot)$, any two of the three identities (55), (65) and (66) imply the third one.*

Theorem 1.5(Polonijo [21]) *Let $(Q, \cdot)$ be a commutative groupoid. Then $(Q, \cdot)$ is palindromic. Furthermore, the constraints (55) and (65) are equivalent, i.e a commutative groupoid $(Q, \cdot)$ is internally medial if and only if it is externally medial.*

Theorem 1.6(Polonijo [21]) *For any quasigroup $(Q,\cdot)$ and $i \in \{1,2,...,6\}$, C_i is valid if and only if the quasigroup is commutative.*

Theorem 1.7(Polonijo [21]) *For any quasigroup $(Q,\cdot)$ and $i \in \{1,2,...,14\}$, CM_i holds if and only if the quasigroup is both commutative and internally medial.*

Theorem 1.8(Polonijo [21]) *For any quasigroup $(Q,\cdot)$ and $i \in \{1,2,...,14\}$, CM_i is valid if and only if the quasigroup is both commutative and externally medial.*

Theorem 1.9(Polonijo [21]) *A quasigroup $(Q,\cdot)$ is palindromic if and only if there exists an automorphism α such that*

$$\alpha(x\cdot y) = y\cdot x \ \forall \ x,y \in Q$$

holds.

It is important to study the characterization of varieties of groupoids and quasigroups represented by linear-bivariate polynomials over the ring $\mathbb{Z}_n$ even though very few of such have been sighted as examples in the past.

§2 Main Results

Theorem 2.1 *Let $P(x,y) = a + bx + cy$ be a linear bivariate polynomial over $\{\mathbb{Z}_n, \mathbb{Z}_p\}$ such that "HYPO" is true. $P(x,y)$ represents a "NAME" {groupoid, quasigroup} $\{(\mathbb{Z}_n, P), (\mathbb{Z}_p, P)\}$ over $\{\mathbb{Z}_n, \mathbb{Z}_p\}$ if and only if "N and S" is true. (Table 1)*

Proof There are 66 identities for which the theorem above is true for in a groupoid or quasigroup. For the sake of space, we shall only demonstrate the proof for one identity for each category.

(**A**) Those identities which involve one element only on each side of the equality sign:

Lemma 2.1 *Let $P(x,y) = a + bx + cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$. $P(x,y)$ represents a unipotent groupoid $(\mathbb{Z}_n, P)$ over $\mathbb{Z}_n$ if and only if $(b+c)(x-y) = 0$ for all $x,y \in \mathbb{Z}_n$.*

Proof $P(x,y)$ satisfies the unipotent law $\Leftrightarrow P(x,x) = P(y,y) \Leftrightarrow a + bx + cx = a + by + cy$ $\Leftrightarrow a + bx - cx - a - by - cy = 0 \Leftrightarrow (b+c)(x-y) = 0$ as required. □

Lemma 2.2 *Let $P(x,y) = a + bx + cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$. $P(x,y)$ represents a unipotent quasigroup $(\mathbb{Z}_n, P)$ over $\mathbb{Z}_n$ if and only if $(b+c)(x-y) = 0$ and $(b,n) = (c,n) = 1$ for all $x,y \in \mathbb{Z}_n$.*

Proof This is proved by using Lemma 2.1 and Theorem 1.2. □

Theorem 2.2 *Let $P(x,y) = a + bx + cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$. $P(x,y)$ represents a unipotent groupoid $(\mathbb{Z}_n, P)$ over $\mathbb{Z}_n$ if and only if $b + c \equiv 0(\mathrm{mod}n)$.*

Proof This is proved by using Lemma 2.1. □

Theorem 2.3 *Let $P(x,y)=a+bx+cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$. $P(x,y)$ represents a unipotent quasigroupp $(\mathbb{Z}_n,P)$ over $\mathbb{Z}_n$ if and only if $b+c\equiv 0(\bmod n)$ and $(b,n)=(c,n)=1$.*

Proof This is proved by using Lemma 2.2. □

Example 2.1 $P(x,y)=5x+y$ is a linear bivariate polynomial over $\mathbb{Z}_6$. $(\mathbb{Z}_6,P)$ is a unipotent groupoid over $\mathbb{Z}_6$.

Example 2.2 $P(x,y)=1+5x+y$ is a linear bivariate polynomial over $\mathbb{Z}_6$. $(\mathbb{Z}_6,P)$ is a unipotent quasigroup over $\mathbb{Z}_6$.

(**B**) Those identities which involve two elements on one or both sides of the equality sign:

Lemma 2.3 *Let $P(x,y)=a+bx+cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$. $P(x,y)$ represents a Stein third groupoid $(\mathbb{Z}_n,P)$ over $\mathbb{Z}_n$ if and only if $a(1+b+c)+x(b^2+c^2)+y(2bc-1)=0$ for all $x,y\in\mathbb{Z}_n$.*

Proof $P(x,y)$ satisfies the Stein third law $\Leftrightarrow P[P(x,y),P(y,x)]=y \Leftrightarrow a(1+b+c)+x(b^2+c^2)+y(2bc-1)=0$ as required. □

Lemma 2.4 *Let $P(x,y)=a+bx+cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$. $P(x,y)$ represents a Stein third quasigroup $(\mathbb{Z}_n,P)$ over $\mathbb{Z}_n$ if and only if $a(1+b+c)+x(b^2+c^2)+y(2bc-1)=0$ and $(b,n)=(c,n)=1$ for all $x,y\in\mathbb{Z}_n$.*

Proof This is proved by using Lemma 2.3 and Theorem 1.2. □

Theorem 2.4 *Let $P(x,y)=a+bx+cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$. $P(x,y)$ represents a Stein third groupoid $(\mathbb{Z}_n,P)$ over $\mathbb{Z}_n$ if and only if $b^2+c^2\equiv 0(\mathrm{mod} n)$, $2bc\equiv 1(\mathrm{mod} n)$ and $a=0$.*

Proof This is proved by using Lemma 2.3. □

Theorem 2.5 *Let $P(x,y)=a+bx+cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$. $P(x,y)$ represents a Stein third quasigroup $(\mathbb{Z}_n,P)$ over $\mathbb{Z}_n$ if and only if $b^2+c^2\equiv 0(\mathrm{mod} n)$, $2bc\equiv 1(\mathrm{mod} n)$ and $a=0$.*

Proof This is proved by using Lemma 2.4. □

Theorem 2.6 *Let $P(x,y)=a+bx+cy$ be a linear bivariate polynomial over $\mathbb{Z}_p$ such that $a\neq 0$. $P(x,y)$ represents a Stein third groupoid $(\mathbb{Z}_p,P)$ over $\mathbb{Z}_p$ if and only if $b^2+c^2\equiv 0(\mathrm{mod} p)$ and $2bc\equiv 1(\mathrm{mod} p)$.*

Proof This is proved by using Lemma 2.3. □

Theorem 2.7 *Let $P(x,y)=a+bx+cy$ be a linear bivariate polynomial over $\mathbb{Z}_p$ such that $a\neq 0$. $P(x,y)$ represents a Stein third quasigroup $(\mathbb{Z}_p,P)$ over $\mathbb{Z}_p$ if and only if $b^2+c^2\equiv 0(\mathrm{mod} p)$ and $2bc\equiv 1(\mathrm{mod} p)$.*

Proof This is proved by using Lemma 2.4. $\square$

Example 2.3 $P(x,y) = 2x + 3y$ is a linear bivariate polynomial over $\mathbb{Z}_5$. $(\mathbb{Z}_5, P)$ is a Stein third groupoid over $\mathbb{Z}_5$.

Example 2.4 $P(x,y) = 2x + 3y$ is a linear bivariate polynomial over $\mathbb{Z}_5$. $(\mathbb{Z}_5, P)$ is a Stein third quasigroup over $\mathbb{Z}_5$.

(**C**) Those identities which involve three distinct elements on one or both sides of the equality sign:

Lemma 2.5 *Let $P(x,y) = a + bx + cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$. $P(x,y)$ represents an Abel-Grassman groupoid $(\mathbb{Z}_n, P)$ over $\mathbb{Z}_n$ if and only if $(x-z)(b-c^2) = 0$ for all $x, z \in \mathbb{Z}_n$.*

Proof $P(x,y)$ satisfies the Abel-Grassman law $\Leftrightarrow P[x, P(y,z)] = P[z, P(y,x)] \Leftrightarrow P(x, a+by+cz) = P(z, a+by+cx) \Leftrightarrow a+bx+c(a+by+cz) = a+bz+c(a+by+cx) \Leftrightarrow (x-z)(b-c^2) = 0$ as required. $\square$

Lemma 2.6 *Let $P(x,y) = a + bx + cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$. $P(x,y)$ represents an Abel-Grassman quasigroup $(\mathbb{Z}_n, P)$ over $\mathbb{Z}_n$ if and only if $(x-z)(b-c^2) = 0$ and $(b,n) = (c,n) = 1$. for all $x, y, z \in \mathbb{Z}_n$.*

Proof This is proved by using Lemma 2.5 and Theorem 1.2. $\square$

Theorem 2.8 *Let $P(x,y) = a + bx + cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$. $P(x,y)$ represents an Abel-Grassman groupoid $(\mathbb{Z}_n, P)$ over $\mathbb{Z}_n$ if and only if $c^2 \equiv b(\text{mod}n)$.*

Proof This is proved by using Lemma 2.5. $\square$

Theorem 2.9 *Let $P(x,y) = a + bx + cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$. $P(x,y)$ represents an Abel-Grassman quasigroup $(\mathbb{Z}_n, P)$ over $\mathbb{Z}_n$ if and only if $c^2 \equiv b(\text{mod}n)$ and $(b,n) = (c,n) = 1$.*

Proof This is proved by using Lemma 2.6. $\square$

Example 2.5 $P(x,y) = 2 + 4x + 2y$ is a linear bivariate polynomial over $\mathbb{Z}_6$. $(\mathbb{Z}_6, P)$ is an Abel-Grassman groupoid over $\mathbb{Z}_6$.

Example 2.6 $P(x,y) = 2 + 4x + 2y$ is a linear bivariate polynomial over $\mathbb{Z}_5$. $(\mathbb{Z}_5, P)$ is an Abel-Grassman quasigroup over $\mathbb{Z}_5$.

(**D**) Those involving four elements:

Lemma 2.7 *Let $P(x,y) = a + bx + cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$. $P(x,y)$ represents an external medial groupoid $(\mathbb{Z}_n, P)$ over $\mathbb{Z}_n$ if and only if $w(b^2-c^2) + z(c^2-b^2) = 0$ for all $w, z \in \mathbb{Z}_n$.*

Proof $P(x,y)$ satisfies the external medial law $\Leftrightarrow P[P(w,x), P(y,z)] = P[P(z,x), P(y,w)]$

$\Leftrightarrow a+b(a+bw+cx)+c(a+by+cz) = a+b(a+bz+cx)+c(a+by+cw) \Leftrightarrow w(b^2-c^2)+z(c^2-b^2) = 0$ as required. $\square$

Lemma 2.8 *Let $P(x,y) = a + bx + cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$. $P(x,y)$ represents an external medial quasigroup $(\mathbb{Z}_n, P)$ over $\mathbb{Z}_n$ if and only if $w(b^2-c^2)+z(c^2-b^2) = 0$ and $(b,n) = (c,n) = 1$ for all $w, z \in \mathbb{Z}_n$.*

Proof This is proved by using Lemma 2.7 and Theorem 1.2. $\square$

Theorem 2.10 *Let $P(x,y) = a + bx + cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$. $(\mathbb{Z}_n, P)$ represents an external medial groupoid over $\mathbb{Z}_n$ if and only if $b^2 \equiv c^2(\mathrm{mod} n)$.*

Proof This is proved by using Lemma 2.7. $\square$

Theorem 2.11 *Let $P(x,y) = a + bx + cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$. $(\mathbb{Z}_n, P)$ represents an external medial quasigroup over $\mathbb{Z}_n$ if and only if $b^2 \equiv c^2(\mathrm{mod} n)$ and $(b,n) = (c,n) = 1$.*

Proof This is proved by using Lemma 2.8 and Theorem 1.2. $\square$

Example 2.7 $P(x,y) = 4 + 2x + 2y$ is a linear bivariate polynomial over $\mathbb{Z}_6$. $(\mathbb{Z}_6, P)$ is an external medial groupoid over $\mathbb{Z}_6$.

Example 2.8 $P(x,y) = 2 + 8x + y$ is a linear bivariate polynomial over $\mathbb{Z}_9$. $(\mathbb{Z}_9, P)$ is an external medial quasigroup over $\mathbb{Z}_9$.

(**E**) Those involving left or right inverse elements:

Lemma 2.9 *Let $P(x,y) = a + bx + cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$. $P(x,y)$ represents a cross inverse property groupoid $(\mathbb{Z}_n, P)$ over $\mathbb{Z}_n$ if and only if $a(bc-1) + x(b^2c + 1 - b - bc) + cy(bc-1) = 0$ for all $x, y \in \mathbb{Z}_n$.*

Proof $P(x,y)$ satisfies the cross inverse property $\Leftrightarrow P[P(x,y), x^\rho)] = y \Leftrightarrow P(a + bx + cy, x^\rho) = y \Leftrightarrow a + b(a + bx + cy) + cx^\rho = y \Leftrightarrow a(bc-1) + x(b^2c + 1 - b - bc) + cy(bc-1) = 0$ as required. $\square$

Lemma 2.10 *Let $P(x,y) = a + bx + cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$. $P(x,y)$ represents a cross inverse property quasigroup $(\mathbb{Z}_n, P)$ over $\mathbb{Z}_n$ if and only if $a(bc-1)+x(b^2c+1-b-bc)+cy(bc-1) = 0$ and $(b,n) = (c,n) = 1$ for all $x, y, z \in \mathbb{Z}_n$.*

Proof This is proved by using Lemma 2.9 and Theorem 1.2. $\square$

Theorem 2.12 *Let $P(x,y) = a + bx + cy$ be a linear bivariate polynomial over $\mathbb{Z}_p$ such that $a \neq 0$. $P(x,y)$ represents a CIP quasigroup $(\mathbb{Z}_p, P)$ over $\mathbb{Z}_p$ if and only if $bc \equiv 1(\mathrm{mod} p)$.*

Proof This is proved by using Lemma 2.10. $\square$

Theorem 2.13 *Let $P(x,y) = a + bx + cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$ such that*

$a \neq 0$ and c is invertible in $\mathbb{Z}_n$. $P(x,y)$ represents a CIP groupoid $(\mathbb{Z}_n, P)$ over $\mathbb{Z}_n$ if and only if $bc \equiv 1(\mathrm{mod} n)$.

Proof This is proved by using Lemma 2.9. □

Theorem 2.14 *Let $P(x,y) = a + bx + cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$ such that $a \neq 0$, c is invertible in $\mathbb{Z}_n$ and $(b,n) = (c,n) = 1$. $P(x,y)$ represents a CIP quasigroup $(\mathbb{Z}_n, P)$ over $\mathbb{Z}_n$ if and only if $bc \equiv 1(\mathrm{mod} n)$.*

Proof This is proved by using Lemma 2.10. □

Theorem 2.15 *Let $P(x,y) = a + bx + cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$. $P(x,y)$ represents a CIP groupoid $(\mathbb{Z}_n, P)$ over $\mathbb{Z}_n$ if $bc \equiv 1(\mathrm{mod} n)$.*

Proof This is proved by using Lemma 2.9. □

Theorem 2.16 *Let $P(x,y) = a + bx + cy$ be a linear bivariate polynomial over $\mathbb{Z}_n$ such that $(b,n) = (c,n) = 1$. $P(x,y)$ represents a CIP quasigroup $(\mathbb{Z}_n, P)$ over $\mathbb{Z}_n$ if $bc \equiv 1(\mathrm{mod} n)$.*

Proof This is proved by using Lemma 2.10. □

Example 2.9 $P(x,y) = 2 + 4x + 4y$ is a linear bivariate polynomial over $\mathbb{Z}_5$. $(\mathbb{Z}_5, P)$ is a cross inverse property groupoid over $\mathbb{Z}_5$.

Example 2.10 $P(x,y) = 3 + 4x + 4y$ is a linear bivariate polynomial over $\mathbb{Z}_5$. $(\mathbb{Z}_5, P)$ is a cross inverse property quasigroup over $\mathbb{Z}_5$.

Table 1. Varieties of groupoids and quasigroups generated by $P(x,y)$ over $\mathbb{Z}_n$

S/N	NAME	G	Q	$\mathbb{Z}_n$	$\mathbb{Z}_p$	HYPO	N AND S	EXAMPLE
1	Idempotent	✓		✓			$b+c=1,\ a=0$	$5x+2y,\ \mathbb{Z}_6$
2	Unipotent	✓		✓			$b+c=0$	$2+4x+2y,\ \mathbb{Z}_6$
			✓	✓			$b+c=0,(b,n)=(c,n)=1$	$2+5x+y,\ \mathbb{Z}_6$
3	Commut	✓		✓			$b=c$	$1+4x+4y,\ \mathbb{Z}_6$
			✓	✓			$b=c,(b,n)=(c,n)=1$	$1+5x+5y,\ \mathbb{Z}_6$
4	Sade Right	✓			✓	$a\neq 0$	$b=-1$	$2+6x+4y,\ \mathbb{Z}_7$
			✓		✓	$a\neq 0$	$b=-1$	$1+5x+4y,\ \mathbb{Z}_7$
5	Sade Left	✓			✓	$a\neq 0$	$c=-1$	$2+4x+5y,\ \mathbb{Z}_7$
			✓		✓	$a\neq 0$	$c=-1$	$2+5x+5y,\ \mathbb{Z}_7$
6	Right	✓			✓	$a\neq 0$	$b=c=1$	$3+x+y,\ \mathbb{Z}_7$
	Alternative		✓		✓	$a\neq 0$	$b=c=1$	$3+x+y,\ \mathbb{Z}_7$
7	Left	✓			✓	$a\neq 0$	$b=c=1$	$2+x+y,\ \mathbb{Z}_7$
	Alternative		✓		✓	$a\neq 0$	$b=c=1$	$2+x+y,\ \mathbb{Z}_7$

S/N	NAME	G	Q	$\mathbb{Z}_n$	$\mathbb{Z}_p$	HYPO	N AND S	EXAMPLE
8	Medial	✓			✓	$a \neq 0$	$b = c$	$2 + 4x + 4y,\ \mathbb{Z}_7$
	Alternative	✓			✓	$b \neq c$	$b + c = 1$	$2 + 4x + 2y,\ \mathbb{Z}_5$
			✓		✓	$a \neq 0$	$b = c$	$2 + 4x + 4y,\ \mathbb{Z}_7$
			✓		✓	$b \neq c$	$b + c = 1$	$2 + 4x + 2y,\ \mathbb{Z}_7$
9	Right	✓			✓	$a \neq 0$	$b = c = -1$	$2 + 4x + 4y,\ \mathbb{Z}_5$
	Semi	✓		✓		$a = 0$	$bc = 1,\ c^2 = -b$	$5x + 2y,\ \mathbb{Z}_9$
	Symmetry		✓		✓	$a \neq 0$	$b = c = -1$	$2 + 4x + 4y,\ \mathbb{Z}_5$
			✓	✓		$a = 0$	$bc = 1,\ c^2 = -b$	$5x + 2y,\ \mathbb{Z}_9$
10	Left	✓			✓	$a \neq 0$	$b = c = -1$	$3 + 4x + 4y,\ \mathbb{Z}_5$
	Semi	✓		✓		$a = 0$	$b = 1,\ b^2 = -c$	$x + 9y,\ \mathbb{Z}_{10}$
	Symmetry		✓		✓	$a \neq 0$	$b = c = -1$	$3 + 4x + 4y,\ \mathbb{Z}_5$
			✓	✓		$a = 0$	$b = 1,\ b^2 = -c$	$x + 9y,\ \mathbb{Z}_{10}$
11	Stein First	✓			✓	$a \neq 0$	$b = c$	$3 + 4x + 4y,\ \mathbb{Z}_5$
			✓		✓	$a \neq 0$	$b = c$	$2 + 4x + 4y,\ \mathbb{Z}_5$
12	Stein	✓			✓	$a \neq 0$	$b = c$	$3 + 4x + 4y,\ \mathbb{Z}_5$
	Second		✓		✓	$a \neq 0$	$b = c$	$2 + 4x + 4y,\ \mathbb{Z}_5$
13	Schroder	✓		✓			$b^2 + c^2 = 1,\ 2bc = a = 0$	$2x + 3y,\ \mathbb{Z}_6$
	Second		✓	✓			$b^2 + c^2 = (b, n) = (c, n) = 1,\ 2bc = a = 0$	?
		✓			✓	$a \neq 0$	$b + c = -1,\ b^2 + c^2 = 1,\ 2bc = 0$	?
			✓		✓	$a \neq 0$	$b + c = -1,\ b^2 + c^2 = 1,\ 2bc = 0$	?
14	Stein Third	✓		✓			$b^2 + c^2 = 0,\ 2bc = 1,\ a = 0$	?
			✓	✓			$(b, n) = (c, n) = 2bc = 1,\ b^2 + c^2 = a = 0$	?
		✓			✓	$a \neq 0$	$b^2 + c^2 = 0,\ 2bc = 1,$	$3 + 2x + 4y,\ \mathbb{Z}_5$
			✓		✓	$a \neq 0$	$b^2 + c^2 = 0,\ 2bc = 1,$	$2 + 2x + 4y,\ \mathbb{Z}_5$
15	Associative	✓			✓	$a \neq 0$	$b = c = 1$	$2 + x + y,\ \mathbb{Z}_6$
			✓		✓	$a \neq 0$	$b = c = 1$	$2 + x + y,\ \mathbb{Z}_6$
16	Slim	✓		✓		$a = 0,\ c$ invert	$bc = 0,\ c = 1$	!
			✓	✓		$a = 0,\ c$ invert	$bc = 0,\ c = 1,\ (b, n) = (c, n) = 1$	?
17	Cyclic	✓		✓			$b = c = 1$	$3 + x + y,\ \mathbb{Z}_6$
	Associativity		✓	✓			$b = c = 1,\ (b, n) = (c, n) = 1$	$3 + x + y,\ \mathbb{Z}_6$
18	Right	✓		✓			$b = 1$	$1 + x + 5y,\ \mathbb{Z}_6$
	Permutability		✓	✓			$b = 1,\ (b, n) = (c, n) = 1$	$1 + x + 5y,\ \mathbb{Z}_6$
19	Left	✓		✓			$c = 1$	$1 + 5x + y,\ \mathbb{Z}_6$
	Permutability		✓	✓			$c = 1,\ (b, n) = (c, n) = 1$	$3 + 5x + y,\ \mathbb{Z}_6$
20	Abel	✓		✓			$c^2 = b$	$2 + 4x + 2y,\ \mathbb{Z}_6$
	Grassman		✓	✓			$c^2 = b,\ (b, n) = (c, n) = 1$	$2 + 4x + 2y,\ \mathbb{Z}_9$
21	Commuting	✓			✓	$a \neq 0$	$b = c = 1$	$1 + x + y,\ \mathbb{Z}_7$
	Product		✓		✓	$a \neq 0$	$b = c = 1$	$1 + x + y,\ \mathbb{Z}_7$
22	Dual Comm	✓			✓	$a \neq 0$	$b = c = 1$	$1 + x + y,\ \mathbb{Z}_7$
	Product		✓		✓	$a \neq 0$	$b = c = 1$	$1 + x + y,\ \mathbb{Z}_7$
23	Right	✓			✓	$a \neq 0$	$b = 1,\ c = -1$	$2 + x + 6y,\ \mathbb{Z}_7$
	Transitivity		✓		✓	$a \neq 0$	$b = 1,\ c = -1$	$2 + x + 6y,\ \mathbb{Z}_7$
24	Left	✓			✓	$a \neq 0$	$b = -1,\ c = 1$	$2 + 6x + y,\ \mathbb{Z}_7$
	Transitivity		✓		✓	$a \neq 0$	$b = -1,\ c = 1$	$2 + 6x + y,\ \mathbb{Z}_7$
25	Schweitzer	✓		✓		$b,\ c$ invert	$b = 1,\ c = -1$	$2 + x + 5y,\ \mathbb{Z}_6$
			✓	✓		$b,\ c$ invert	$b = 1,\ c = -1,\ (b, n) = (c, n) = 1$	$2 + x + 5y,\ \mathbb{Z}_6$
		✓			✓	$a \neq 0$	$b = 1,\ c = -1$	$3 + x + 6y,\ \mathbb{Z}_7$
			✓		✓	$a \neq 0$	$b = 1,\ c = -1$	$3 + x + 6y,\ \mathbb{Z}_7$
26	Dual of	✓		✓		$b,\ c$ invert	$b = 1,\ c = -1$	$2 + x + 5y,\ \mathbb{Z}_6$
	Schweitzer		✓	✓		$b,\ c$ invert	$b = 1,\ c = -1,\ (b, n) = (c, n) = 1$	$2 + x + 5y,\ \mathbb{Z}_6$
		✓			✓	$a \neq 0$	$b = 1,\ c = -1$	$3 + x + 6y,\ \mathbb{Z}_7$
			✓		✓	$a \neq 0$	$b = 1,\ c = -1$	$3 + x + 6y,\ \mathbb{Z}_7$

S/N	NAME	G	Q	$\mathbb{Z}_n$	$\mathbb{Z}_p$	HYPO	N AND S	EXAMPLE
27	Right Self	✓			✓		$c = 1 - b,\ a = 0$	$3x + 5y,\ \mathbb{Z}_7$
	Distributive		✓		✓		$c = 1 - b,\ a = 0$	$3x + 5y,\ \mathbb{Z}_7$
28	Left Self	✓			✓		$c = 1 - b,\ a = 0$	$3x + 5y,\ \mathbb{Z}_7$
	Distributive		✓		✓		$c = 1 - b,\ a = 0$	$3x + 5y,\ \mathbb{Z}_7$
29	Right	✓		✓		$b,\ c$ invert	$b = c,\ 2b^2 = b$	?
	Abelian		✓	✓		$b,\ c$ invert	$b = c,\ 2b^2 = b$	?
	Distributivity	✓		✓		$a \neq 0$	$b = c,\ 2b^2 = b$	?
			✓	✓		$a \neq 0$	$b = c,\ 2b^2 = b$	?
		✓			✓	$a \neq 0$	$b = c,\ 2b = 1$	$2 + 3x + 3y,\ \mathbb{Z}_5$
			✓		✓	$a \neq 0$	$b = c,\ 2b = 1$	$2 + 3x + 3y,\ \mathbb{Z}_5$
30	Left	✓		✓		$b,\ c$ invert	$b = c,\ 2b^2 = b$	
	Abelian		✓	✓		$b,\ c$ invert	$b = c,\ 2b^2 = b$	?
	Distributivity	✓		✓		$a \neq 0$	$b = c,\ 2b^2 = b$	?
			✓	✓		$a \neq 0$	$b = c,\ 2b^2 = b$	?
		✓			✓	$a \neq 0$	$b = c,\ 2b = 1$	$2 + 3x + 3y,\ \mathbb{Z}_5$
			✓		✓	$a \neq 0$	$b = c,\ 2b = 1$	$2 + 3x + 3y,\ \mathbb{Z}_5$
31	Bol	✓		✓			$b = c = 1$	$2 + x + y,\ \mathbb{Z}_6$
	Moufang		✓	✓		$(b, n) = (c, n) = 1$	$b = c = 1$	$2 + x + y,\ \mathbb{Z}_6$
32	Dual Bol	✓		✓			$b = c = 1$	$2 + x + y,\ \mathbb{Z}_6$
	Moufang		✓	✓		$(b, n) = (c, n) = 1$	$b = c = 1$	$2 + x + y,\ \mathbb{Z}_6$
33	Moufang	✓			✓		$b = c = 1,\ a = 0$	$x + y,\ \mathbb{Z}_5$
			✓		✓		$b = c = 1,\ a = 0$	$x + y,\ \mathbb{Z}_5$
34	R Bol	✓			✓	$a \neq 0$	$b^2 = 1,\ b = c = 1$	$2 + x + y,\ \mathbb{Z}_7$
			✓		✓	$a \neq 0$	$b^2 = 1,\ b = c = 1$	$2 + x + y,\ \mathbb{Z}_7$
		✓			✓	$-1 \neq b \neq c$	$b^2 = 1, c = 1, a = 0$	$8x + y,\ \mathbb{Z}_{63}$
			✓		✓	$-1 \neq b \neq c$	$b^2 = 1, c = 1, a = 0$	$8x + y,\ \mathbb{Z}_{63}$
35	L Bol	✓			✓	$a \neq 0$	$c^2 = 1,\ b = c = 1$	$2 + x + y,\ \mathbb{Z}_7$
			✓		✓	$a \neq 0$	$c^2 = 1,\ b = c = 1$	$2 + x + y,\ \mathbb{Z}_7$
		✓			✓	$-1 \neq b \neq c$	$c^2 = 1, b = 1, a = 0$	$x + 8y,\ \mathbb{Z}_{63}$
			✓		✓	$-1 \neq b \neq c$	$c^2 = 1, b = 1, a = 0$	$x + 8y,\ \mathbb{Z}_{63}$
36	RC_4	✓			✓	$a = 0$	$c = b^2 = 1$	$8x + y,\ \mathbb{Z}_{63}$
			✓		✓	$a = 0$	$c = b^2 = 1$	$8x + y,\ \mathbb{Z}_{63}$
		✓		✓		$a = 0,\ b, c$ invert	$c = b^2 = 1$	$8x + y,\ \mathbb{Z}_{63}$
			✓	✓		$a = 0,\ b, c$ invert	$c = b^2 = (b, n) = (c, n) = 1$	$8x + y,\ \mathbb{Z}_{63}$
		✓		✓			$b = -1,\ c = 1$	$2 + 5x + y,\ \mathbb{Z}_6$
			✓	✓			$b = -1,\ c = (b, n) = (c, n) = 1$	$2 + 5x + y,\ \mathbb{Z}_6$
37	LC_4	✓			✓	$a = 0$	$b = c^2 = 1$	$x + 8y,\ \mathbb{Z}_{63}$
			✓		✓	$a = 0$	$b = c^2 = 1$	$x + 8y,\ \mathbb{Z}_{63}$
		✓		✓		$a = 0,\ b, c$ invert	$b = c^2 = 1$	$x + 3y,\ \mathbb{Z}_8$
			✓	✓		$a = 0,\ b, c$ invert	$b = c^2 = (b, n) = (c, n) = 1$	$x + 4y,\ \mathbb{Z}_{15}$
		✓		✓			$b = -1,\ c = 1$	$2 + 5x + y,\ \mathbb{Z}_6$
			✓	✓			$b = -1,\ c = (b, n) = (c, n) = 1$	$2 + 5x + y,\ \mathbb{Z}_6$
38	RC_1	✓			✓	$a = 0$	$c = b^2 = 1$	$8x + y,\ \mathbb{Z}_{63}$
			✓		✓	$a = 0$	$c = b^2 = 1$	$8x + y,\ \mathbb{Z}_{63}$
		✓		✓		$a = 0,\ b, c$ invert	$c = b^2 = 1$	$8x + y,\ \mathbb{Z}_{63}$
			✓	✓		$a = 0,\ b, c$ invert	$c = b^2 = (b, n) = (c, n) = 1$	$8x + y,\ \mathbb{Z}_{63}$
		✓		✓			$b = -1,\ c = 1$	$2 + 5x + y,\ \mathbb{Z}_6$
			✓	✓			$b = -1,\ c = (b, n) = (c, n) = 1$	$2 + 5x + y,\ \mathbb{Z}_6$

S/N	NAME	G	Q	$\mathbb{Z}_n$	$\mathbb{Z}_p$	HYPO	N AND S	EXAMPLE
39	LC_1	✓			✓	$a=0,\ c\neq 1$	$c=-1$	$3x+6y,\ \mathbb{Z}_7$
			✓		✓	$a=0,\ c\neq 1$	$c=-1$	$3x+6y,\ \mathbb{Z}_7$
		✓		✓		$a=0,\ c\neq 1,\ c$ invert	$c=-1$	$5x+5y,\ \mathbb{Z}_6$
			✓	✓		$a=0,\ c\neq 1,\ c$ invert	$c=-1,\ (b,n)=(c,n)=1$	$5x+5y,\ \mathbb{Z}_6$
40	LC_3	✓		✓			$c=1,\ b=-2$	$3+4x+y,\ \mathbb{Z}_6$
			✓	✓			$c=1,\ b=-2,\ (b,n)=(c,n)=1$	$2+5x+y,\ \mathbb{Z}_7$
41	RC_3	✓		✓			$c=1,\ b=-2$	$3+4x+y,\ \mathbb{Z}_6$
			✓	✓			$c=1,\ b=-2,\ (b,n)=(c,n)=1$	$2+5x+y,\ \mathbb{Z}_7$
42	C-Law	✓			✓	$a=0$	$b=c=-1$	$4x+4y,\ \mathbb{Z}_5$
			✓		✓	$a=0$	$b=c=-1$	$4x+4y,\ \mathbb{Z}_5$
		✓		✓		$a\neq 0,\ b\neq 1,\ b,c$ inv	$b=c=-1$	$3+5x+5y,\ \mathbb{Z}_6$
			✓	✓		$a\neq 0,\ b\neq 1,\ b,c$ inv	$b=c=-1,\ (b,n)=(c,n)=1$	$3+5x+5y,\ \mathbb{Z}_6$
43	LIP	✓			✓	$a\neq 0$	$c^2=b^2=bc=1$	?
			✓		✓	$a\neq 0$	$c^2=b^2=bc=1$	?
44	RIP	✓			✓	$a\neq 0$	$c^2=b^2=bc=1$	?
			✓		✓	$a\neq 0$	$c^2=b^2=bc=1$	?
45	1st Right	✓			✓	$a\neq 0$	$bc=1$	$2+3x+4y,\ \mathbb{Z}_{11}$
	CIP		✓		✓	$a\neq 0$	$bc=1$	$2+3x+4y,\ \mathbb{Z}_{11}$
		✓		✓		$a\neq 0,\ c$ inv	$bc=1$	$3+3x+3y,\ \mathbb{Z}_8$
			✓	✓		$a\neq 0,\ c$ inv	$bc=1,\ (b,n)=(c,n)=1$	$3+3x+3y,\ \mathbb{Z}_8$
46	2nd Right	✓		✓			$bc=1$	$3+3x+3y,\ \mathbb{Z}_8$
	CIP		✓	✓			$bc=1,\ (b,n)=(c,n)=1$	$3+3x+3y,\ \mathbb{Z}_8$
47	1st Left	✓			✓	$a\neq 0$	$bc=1$	$2+3x+4y,\ \mathbb{Z}_{11}$
	CIP		✓		✓	$a\neq 0$	$bc=1$	$2+3x+4y,\ \mathbb{Z}_{11}$
		✓		✓		$a\neq 0,\ b$ inv	$bc=1$	$3+3x+3y,\ \mathbb{Z}_8$
			✓	✓		$a\neq 0,\ b$ inv	$bc=1,\ (b,n)=(c,n)=1$	$3+3x+3y,\ \mathbb{Z}_8$
48	2nd Left	✓		✓			$bc=1$	$3+3x+3y,\ \mathbb{Z}_8$
	CIP		✓	✓			$bc=1,\ (b,n)=(c,n)=1$	$3+3x+3y,\ \mathbb{Z}_8$
49	R AAIP	✓			✓	$bc+b\neq 1$	$b=c$	$2+4x+4y,\ \mathbb{Z}_{11}$
			✓		✓	$bc+b\neq 1$	$b=c$	$2+4x+4y,\ \mathbb{Z}_{11}$
		✓			✓	$c\neq b$	$b+bc=1$	$2+3x+y,\ \mathbb{Z}_5$
			✓		✓	$c\neq b$	$b+bc=1$	$2+3x+y,\ \mathbb{Z}_5$
50	L AAIP	✓			✓	$bc+b\neq 1$	$b=c$	$2+4x+4y,\ \mathbb{Z}_{11}$
			✓		✓	$bc+b\neq 1$	$b=c$	$2+4x+4y,\ \mathbb{Z}_{11}$
		✓			✓	$c\neq b$	$b+bc=1$	$2+3x+y,\ \mathbb{Z}_5$
			✓		✓	$c\neq b$	$b+bc=1$	$2+3x+y,\ \mathbb{Z}_5$
51	R AIP	✓		✓				$a+bx+cy,\ \mathbb{Z}_n$
			✓	✓				$a+bx+cy,\ \mathbb{Z}_n$
		✓			✓			$a+bx+cy,\ \mathbb{Z}_n$
			✓		✓			$a+bx+cy,\ \mathbb{Z}_n$
52	L AIP	✓		✓				$a+bx+cy,\ \mathbb{Z}_n$
			✓	✓				$a+bx+cy,\ \mathbb{Z}_n$
		✓			✓			$a+bx+cy,\ \mathbb{Z}_n$
			✓		✓			$a+bx+cy,\ \mathbb{Z}_n$
53	R SAIP	✓		✓				$a+bx+cy,\ \mathbb{Z}_n$
			✓	✓				$a+bx+cy,\ \mathbb{Z}_n$
		✓			✓			$a+bx+cy,\ \mathbb{Z}_n$
			✓		✓			$a+bx+cy,\ \mathbb{Z}_n$

S/N	NAME	G	Q	$\mathbb{Z}_n$	$\mathbb{Z}_p$	HYPO	N AND S	EXAMPLE
54	L SAIP	✓		✓				$a+bx+cy,\ \mathbb{Z}_n$
			✓	✓				$a+bx+cy,\ \mathbb{Z}_n$
		✓			✓			$a+bx+cy,\ \mathbb{Z}_n$
			✓		✓			$a+bx+cy,\ \mathbb{Z}_n$
55	R WIP	✓			✓	$a=0,\ c^2\neq 0$	$bc=1$	$3x+5y,\ \mathbb{Z}_7$
			✓		✓	$a=0,\ c^2\neq 0$	$bc=1$	$3x+5y,\ \mathbb{Z}_7$
		✓		✓		$a=0,\ c$ inv	$bc=1$	$3x+4y,\ \mathbb{Z}_6$
			✓	✓		$a=0,\ c$ inv	$bc=1,\ (b,n)=(c,n)=1$	?
		✓		✓		$a=0,\ bc+b\neq 1$	$bc=1$	?
			✓	✓		$a=0,\ bc+b\neq 1$	$bc=1,\ (b,n)=(c,n)=1$	?
56	L WIP	✓			✓	$a=0,\ b^2\neq 0$	$bc=1$	$3x+5y,\ \mathbb{Z}_7$
			✓		✓	$a=0,\ b^2\neq 0$	$bc=1$	$3x+5y,\ \mathbb{Z}_7$
		✓		✓		$a=0,\ b$ inv	$bc=1$	$3x+4y,\ \mathbb{Z}_6$
			✓	✓		$a=0,\ b$ inv	$bc=1,\ (b,n)=(c,n)=1$	?
		✓		✓		$a=0,\ bc+c\neq 1$	$bc=1$	?
			✓	✓		$a=0,\ bc+c\neq 1$	$bc=1,\ (b,n)=(c,n)=1$	?
57	$\mathrm{E_l}$	✓	✓					$a+bx+cy,\ \mathbb{Z}_n$
			✓	✓				$a+bx+cy,\ \mathbb{Z}_n$
		✓			✓			$a+bx+cy,\ \mathbb{Z}_n$
			✓		✓			$a+bx+cy,\ \mathbb{Z}_n$
58	$\mathrm{E_r}$	✓	✓					$a+bx+cy,\ \mathbb{Z}_n$
			✓	✓				$a+bx+cy,\ \mathbb{Z}_n$
		✓			✓			$a+bx+cy,\ \mathbb{Z}_n$
			✓		✓			$a+bx+cy,\ \mathbb{Z}_n$
59	Right F	✓	✓					$a+bx+cy,\ \mathbb{Z}_n$
			✓	✓				$a+bx+cy,\ \mathbb{Z}_n$
		✓			✓			$a+bx+cy,\ \mathbb{Z}_n$
			✓		✓			$a+bx+cy,\ \mathbb{Z}_n$
60	Left F	✓	✓					$a+bx+cy,\ \mathbb{Z}_n$
			✓	✓				$a+bx+cy,\ \mathbb{Z}_n$
		✓			✓			$a+bx+cy,\ \mathbb{Z}_n$
			✓		✓			$a+bx+cy,\ \mathbb{Z}_n$
61	Medial	✓	✓					$a+bx+cy,\ \mathbb{Z}_n$
			✓	✓				$a+bx+cy,\ \mathbb{Z}_n$
		✓			✓			$a+bx+cy,\ \mathbb{Z}_n$
			✓		✓			$a+bx+cy,\ \mathbb{Z}_n$
62	Specialized	✓	✓					$a+bx+cy,\ \mathbb{Z}_n$
	Medial		✓	✓				$a+bx+cy,\ \mathbb{Z}_n$
		✓			✓			$a+bx+cy,\ \mathbb{Z}_n$
			✓		✓			$a+bx+cy,\ \mathbb{Z}_n$
63	First	✓			✓		$b=c$	$2+4x+4y,\ \mathbb{Z}_7$
	Rectangle		✓		✓		$b=c$	$2+4x+4y,\ \mathbb{Z}_7$
		✓		✓		c inv	$b=c$	$2+4x+4y,\ \mathbb{Z}_6$
			✓	✓		c inv	$b=c,\ (b,n)=(c,n)=1$	$2+4x+4y,\ \mathbb{Z}_6$
64	Second	✓			✓		$b=-c$	$2+4x+4y,\ \mathbb{Z}_7$
	Rectangle		✓		✓		$b=-c$	$2+4x+4y,\ \mathbb{Z}_7$
		✓		✓		b inv	$b=-c$	$2+4x+4y,\ \mathbb{Z}_6$
			✓	✓		b inv	$b=-c,\ (b,n)=(c,n)=1$	$2+4x+4y,\ \mathbb{Z}_6$
65	$\mathrm{C}_i,\ i=1-6$		✓		✓		$b=c$	$3+5x+5y,\ \mathbb{Z}_7$
66	$\mathrm{CM}_i,\ i=1-14$		✓		✓	$b\neq -c$	$b=c$	$3+5x+5y,\ \mathbb{Z}_7$

Remark 2.1 A summary of the results on the characterization of groupoids and quasigroups generated by $P(x, y)$ is exhibited in Table 1. In this table, G stands for *groupoid*, Q stands for *quasigroup*, HYPO stands for *hypothesis*, N AND S stands for *necessary and sufficient condition(s)*. Cells with question marks mean examples could not be gotten.

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New Characterizations for Bertrand Curves in Minkowski 3-Space

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Abstract: Bertrand curves have been investigated in Lorentzian and Minkowski spaces and some characterizations have been given in [1,2,6]. In this paper, we have investigated the relations between Frenet vector fields and curvatures and torsions of Bertrand curves at the corresponding points in Minkowski 3-space.

Key Words: Bertrand curves, constant curvature and torsion, Minkowski 3- Space.

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§1. Introduction

In the study of the fundamental theory and the characterizations of space curves, the corresponding relations between the curves are the very interesting and important problem. The well-known Bertrand curve is characterized as a kind of such corresponding relation between the two curves. J. Bertrand studied curves in Euclidean 3-space whose principal normals are the principal normals of another curve. Such a curve is nowadays called a Bertrand curve. Bertrand curves have a characteristic property that curvature and torsion are in linear relation.In the recent work [2], the authors studied spacelike and timelike Bertrand curves in Minkowski 3-space. (See also independently obtained results by [6]).

In this paper, we have investigated the relations between Frenet vector fields and curvatures and torsions of Bertrand curves at the corresponding points in Minkowski 3-space.

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§2. Preliminaries

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the standard flat metric given by

$$\langle,\rangle = -dx_1 + dx_3 + dx_3$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. Since $\langle,\rangle$ is an indefinite metric, recall that a vector $v \in E_1^3$ can have one of three Lorentzian causal characters: it can be spacelike if $\langle v, v\rangle > 0$ or $v = 0$, timelike if $\langle v, v\rangle < 0$ and null (lightlike) if $\langle v, v\rangle = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null (lightlike).

Minkowski space is originally from the relativity in Physics. In fact, a timelike curve corresponds to the path of an observer moving at less than the speed of light. Denote by $\{T, N, B\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space E_1^3. For an arbitrary curve $\alpha(s)$ in the space E_1^3, the following Frenet formulae are given. If α is timelike curve, then the Frenet formulae read

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{1.1}$$

where $\langle T, T\rangle_L = -1, \langle N, N\rangle_L = 1, \langle B, B\rangle_L = 1, \langle T, N\rangle_L = \langle N, B\rangle_L = \langle T, B\rangle_L = 0$. If α is a spacelike curve with a spacelike principal normal, then the Frenet formulae read

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{1.2}$$

where $\langle T, T\rangle_L = \langle N, N\rangle_L = 1, \langle B, B\rangle_L = -1, \langle T, N\rangle_L = \langle N, B\rangle_L = \langle T, B\rangle_L = 0$.If α is a spacelike curve with a spacelike binormal, then the Frenet formulae read

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{1.3}$$

where $\langle T, T\rangle_L = \langle B, B\rangle_L = 1, \langle N, N\rangle_L = -1, \langle T, N\rangle_L = \langle N, B\rangle_L = \langle T, B\rangle_L = 0$ [4,7,11].

§3. Bertrand Curves in Minkowski 3-Space

Definition 3.1([1,2,6]) *Let β_1 and β_2 be two unit speed regular curves in E_1^3, and $\{T_1, N_1, B_1\}$ and $\{T_2, N_2, B_2\}$also be Frenet Frames on these curves, respectively. β_1 and β_2 are called Bertrand curves if N_1 and N_2 are linearly dependent.We say that β_2 is a Bertrand mate for β_1 and β_2 are Bertrand curves. And (β_1, β_2) is called a Bertrand couple and we can write*

$$\beta_2(s) = \beta_1(s) + rN_1(s).$$

Theorem 3.1 *If there exists a one-to-one correspondence between the points of the spacelike curves C_1 and C_2 with timelike principal normal, such that at corresponding points P_1 on C_1 and P_2 on C_2, then the following statements hold:*

(1) *The curvature κ_1 of C_1 is a constant;*
(2) *The torsion τ_2 of C_2 is constant;*
(3) *The unit tangent vector T_1 of C_1 is parallel to the unit tangent vector T_2 of C_2.*

Then the curve C generated by P that divides the segment P_1P_2 in ratio $m:1$ is a spacelike Bertrand curve with timelike principal normal.

Proof We shall use the subscripts 1, 2 to designate the geometric quantites corresponding to the curves C_1, C_2 while the same letters without subscripts will refer to the spacelike curve C with timelike principal normal.

Let $\alpha(s), \alpha_1(s), \alpha_2(s)$ be the coordinat vectors at the points P, P_1, P_2 on the curves C,C_1,C_2 respectively. Then the from convex combination of points P_1 and P_2 , the equation of point P is

$$\alpha(s) = m\alpha_1(s) + (1-m)\alpha_2(s), \ m \in R, \tag{2.1}$$

while by hypothesis,

$$\|T_1\| = \|T_2\| = 1, \quad T_1 = T_2. \tag{2.2}$$

On differentiating Eq.(2.1) we have

$$Tds = mT_1ds_1 + (1-m)T_2ds_2 = (mds_1 + (1-m)ds_2)\, T_1 \tag{2.3}$$

which shows that T is parallel to T_1 and T_2 and always can be chosen so that

$$T = T_1 = T_2, \tag{2.4}$$

and

$$ds = mds_1 + (1-m)ds_2. \tag{2.5}$$

Differentiating of Eq.(2.4) gives

$$\kappa N ds = \kappa_1 N_1 ds_1 = \kappa_2 N_2 ds_2, \tag{2.6}$$

and if we assume that $\kappa, \kappa_1, \kappa_2$ are positive, then

$$N = N_1 = N_2, \tag{2.7}$$

and

$$\kappa ds = \kappa_1 ds_1 = \kappa_2 ds_2. \tag{2.8}$$

From Eq.(2.4) and Eq.(2.7)

$$B = B_1 = B_2, \tag{2.9}$$

and differentiating

$$\tau N ds = \tau_1 N_1 ds_1 = \tau_2 N_2 ds_2. \tag{2.10}$$

Elimination of ds, ds_1, ds_2 gives

$$\left(\frac{m}{\kappa_1}\right)\kappa+\left(\frac{1-m}{\tau_2}\right)\tau=1;\kappa_1\neq 0,\tau_2\neq 0,$$

which is the desired result, since m,κ_1,τ_2 are constant. If instead of $T_1=T_2$ were given the condition $B_1=B_2$, the same result would follow in the same manner. □

Theorem 3.2 *If condition (c) of Theorem 3.1 is modifed so that at corresponding points P_1 and P_2, the binormals B_1 and B_2 are parallel, then the curve C is a spacelike Bertrand curve with timelike principal normal.*

Proof Since $B_1=B_2$ then

$$\tau_1N_1=\tau_2N_2\frac{ds_2}{ds_1}. \tag{2.11}$$

where N_1 and N_2 are the unit normal vectors of α_1 and α_2 at the points P_1 and P_2 with arc-length parametrization. Hence $N_1=N_2$ and $T_1=N_1\times B_1=N_2\times B_2$, we know T_1 parallel to T_2. By Theorem 2.1, C is a spacelike Bertrand curve with timelike principal normal. □

Theorem 3.3 *If condition (c) of Theorem 3.1 is modified so that at corresponding points P_1 and P_2 the tangent at P_1 is parallel to the binormal B_2 at P_2, then the curve C is a spacelike Bertrand curve with timelike principal normal.*

Proof Since $T_1=B_2$, it follow that

$$\kappa_1N_1=\tau_2N_2\frac{ds_2}{ds_1}. \tag{2.12}$$

Hence N_1 is parallel to N_2 and since N_1 and N_2 are unit vectors,

$$N_1=N_2 \tag{2.13}$$

and

$$\frac{ds_2}{ds_1}=\frac{\kappa_1}{\tau_2}, \tag{2.14}$$

since $B_1=T_1\times N_1=B_2\times N_2=-T_2$,we have

$$B_1=-T_2. \tag{2.15}$$

Let α,α_1,α_2 be the coordinate vectors at the points P, P_1, P_2 on the curves C,C_1,C_2, respectively. Then

$$\alpha=m\alpha_1+(1-m)\alpha_2 \tag{2.16}$$

$$\begin{aligned}
\frac{d\alpha}{ds}&=m\frac{d\alpha_1}{ds}+(1-m)\frac{d\alpha_2}{ds}\\
\frac{d\alpha}{ds}&=m\frac{d\alpha_1}{ds_1}\frac{ds_1}{ds}+(1-m)\frac{d\alpha_2}{ds_2}\frac{ds_2}{ds}\\
T&=\left(mT_1+(1-m)\frac{ds_2}{ds_1}T_2\right)\frac{ds_1}{ds}\\
&=\left(mT_1+\frac{\kappa_1}{\tau_2}(1-m)T_2\right)\frac{ds_1}{ds}
\end{aligned}$$

$$T = m_1 T_1 + m_2 T_2 \ , \tag{2.17}$$

where

$$\begin{aligned} m_1 &= m\frac{ds_1}{ds} = \frac{m \ \tau_2}{\sqrt{(m \ \tau_2)^2 + [\kappa_1(1-m)]^2}}, \ m_1 = const. \\ m_2 &= (1-m)\frac{ds_2}{ds} = \frac{(1-m) \ \kappa_1}{\sqrt{(m \ \tau_2)^2 + [\kappa_1(1-m)]^2}}, m_2 = const. \end{aligned}$$

Differentiating Eq.(2.17), one gets

$$\kappa N = \kappa_1 m_1 N_1 \frac{ds_1}{ds} = \kappa_2 m_2 N_2 \frac{ds_2}{ds}. \tag{2.18}$$

Hence

$$N = N_1 = N_2 \tag{2.19}$$

and

$$\kappa = \kappa_1 m_1 \frac{ds_1}{ds} + \kappa_2 m_2 \frac{ds_2}{ds}. \tag{2.20}$$

Using Eq.(2.7) and Eq.(2.9), one finds that

$$B = m_1 B_1 + m_2 B_2. \tag{2.21}$$

Differentiating Eq.(2.11), one gets

$$\tau N = \tau_1 m_1 \frac{ds_1}{ds} N_1 + \tau_2 m_2 \frac{ds_2}{ds} N_2. \tag{2.22}$$

Hence

$$\tau = \tau_1 m_1 \frac{ds_1}{ds} + \tau_2 m_2 \frac{ds_2}{ds}. \tag{2.23}$$

Using Eq.(2.14) and Eq.(2.15), one gets

$$\frac{ds_2}{ds_1} = -\frac{\tau_1}{\kappa_2} = \frac{\kappa_1}{\tau_2}. \tag{2.24}$$

and

$$\frac{\tau_1}{\kappa_1} = -\frac{\kappa_2}{\tau_2}.$$

Let

$$M_1 = m_1 \frac{ds_1}{ds}, \qquad M_2 = m_2 \frac{ds_2}{ds}.$$

Then using Eq.(2.20) and Eq.(2.23), one gets

$$\begin{aligned} \frac{\kappa}{M_2 \tau_2} + \frac{\tau}{M_1 \kappa_1} &= \frac{\kappa_1}{\tau_2}\frac{M_1}{M_2} + \left(\frac{\kappa_2 M_2}{\tau_2 M_2} + \frac{\tau_1 M_1}{\kappa_1 M_1}\right) + \frac{\tau_2}{\kappa_1}\frac{M_2}{M_1} \\ &= \frac{\kappa_1}{\tau_2}\frac{M_1}{M_2} + \frac{\tau_2}{\kappa_1}\frac{M_2}{M_1} = \text{constant}, \quad \frac{\kappa_1}{\tau_2}, \ \frac{M_1}{M_2} = \text{constant}. \end{aligned}$$

and this is the intrinsic equation of a spacelike Bertrand curve. □

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Respectable Graphs

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Abstract: Two graphs X and Y are said to be respectable to each other if $\mathcal{A}(X) = \mathcal{A}(Y)$. In this study we explore some graph theoretic and algebraic properties shared by the respectable graphs.

Key Words: Adjacency algebra, coherent algebra, walk regular graphs, vertex transitive graphs.

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§1. Introduction

Let $A(X)$ (or simply A, if X is clear from the context) be the adjacency matrix of a graph X. The set of all polynomials in A with coefficients from the field of complex numbers $\mathbb{C}$ forms an algebra called the adjacency algebra of X, denoted by $\mathcal{A}(X)$. Let $\dim(\mathcal{A}(X))$ denote the dimension of $\mathcal{A}(X)$ as a vector space over $\mathbb{C}$. It is easy to see that $\dim(\mathcal{A}(X))$ is equal to degree of the minimal polynomial of A. Since $\dim(\mathcal{A}(X))$ is also equal $|spec(A)|$ where $spec(A)$ denote the set of all distinct eigenvalues of A and $|B|$ denote the cardinality of the set B.

Definition 1 *Two graphs X and Y are said to be respectable to each other if $\mathcal{A}(X) = \mathcal{A}(Y)$. In this case we say that either X respects Y or Y respects X.*

A graph Y is said to be a polynomial in a graph X if $A(Y) \in \mathcal{A}(X)$. For example, K_n the complete graph is a polynomial in every connected regular graph with n vertices. By definition if X respects Y, then X is a polynomial in Y and Y is a polynomial in X. In this study we explore some graph theoretic and algebraic properties shared by respectable graphs. In the remaining part of this section we will give some preliminaries required for this paper.

For two vertices u and v of a connected graph X, let $d(u,v)$ denote the length of the shortest path from u to v. Then the diameter of a connected graph $X = (V,E)$ is $\max\{d(u,v) : u,v \in V\}$. It is shown in Biggs [3] that if X is a connected graph with diameter ℓ, then $\ell + 1 \le \dim(\mathcal{A}(X)) \le n$.

A graph $X_1 = (V(X_1), E(X_1))$ is said to be *isomorphic* to a graph $X_2 = (V(X_2), E(X_2))$, written $X_1 \cong X_2$, if there is a one-to-one correspondence $\rho : V(X_1) \to V(X_2)$ such that $\{v_1, v_2\} \in E(X_1)$ if and only if $\{\rho(v_1), \rho(v_2)\} \in E(X_2)$. In such a case, ρ is called an *isomor-*

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phism of X_1 and X_2. An isomorphism of a graph X onto itself is called an *automorphism*. The collection of all automorphisms of a graph X is denoted by $\mathrm{Aut}(X)$. It is well known that $\mathrm{Aut}(X)$ is a group under composition of two maps. It is easy to see that $\mathrm{Aut}(X) = \mathrm{Aut}(X^c)$, where X^c is the complement of the graph X. If X is a graph with n vertices we can think $\mathrm{Aut}(X)$ as a subgroup of S_n. Under this correspondence, if a graph X has n vertices then $\mathrm{Aut}(X)$ consists of $n \times n$ permutation matrices and for each $g \in \mathrm{Aut}(X)$, P_g will denote the corresponding permutation matrix.

§2. Graph Theoretic Properties

In this section we will see some graph theoretical properties shared by the respectable graphs. The next result gives a method to check whether a given permutation matrix is an element of $\mathrm{Aut}(X)$ or not.

Lemma 2.1(Biggs [3]) *Let A be the adjacency matrix of a graph X. Then $g \in Aut(X)$ is an automorphism of X if and only if $P_gA = AP_g$.*

The following result is immediate from the above lemma, also given by Paul.M.Weichsel [7].

Corollary 2.2 *Let X be a graph and $p(x)$ be a polynomial such that $p(X)$ is a graph. Then $Aut(X) \subseteq Aut(p(X))$.*

Corollary 2.3 *If the graph X respects the graph Y, then $Aut(X) = Aut(Y)$.*

Lemma 2.4(Biggs [3]) *A graph X is regular if and only if $A(X)J = JA(X)$, where J is a matrix with each entry is 1.*

The following result shows that any graph which is a polynomial in a regular graph is regular.

Corollary 2.5 *Let X be a regular graph. The any graph which is a polynomial in X is also regular . In particular if X respects Y, then Y is regular.*

Lemma 2.6(Biggs [3]) *A graph X is connected regular if and only if $J \in \mathcal{A}(X)$.*

Corollary 2.7 *If X is a regular graph then J is polynomial in either A or A^c.*

Proof For every graph X, either X or X^c is connected. Hence the result follows from the above lemma. □

Corollary 2.8 *Let X be a connected regular graph, then X^c is connected if and only if X respects X^c.*

Proof It is easy to verify that X^c is also regular. Since X is connected regular graph from Lemma 2.6 we have $J \in \mathcal{A}(X) \Rightarrow A(X^c) = J - I - A \in \mathcal{A}(X) \Rightarrow \mathcal{A}(X^c) \subseteq \mathcal{A}(X)$. Now it is

sufficient to prove that X^c is connected if and only if $\mathcal{A}(X) \subseteq \mathcal{A}(X^c)$.
X^c is connected $\Leftrightarrow J \in \mathcal{A}(X^c) \Leftrightarrow A \in \mathcal{A}(X^c) \Leftrightarrow \mathcal{A}(X) \subseteq \mathcal{A}(X^c)$. $\square$

Corollary 2.9 *Let X be a connected regular graph. If X respects Y, then Y is connected regular graph.*

We say that a graph X is walk-regular if, for each s, the number of closed walks of length s starting at a vertex v is independent of the choice of v.
Theorem 2.10([6]) *Let A be the adjacency matrix of a graph X. Then X is walk-regular if and only if the diagonal entries of A^s $\forall s$ are all equal.*

Corollary 2.11 *Let X be a walk regular graph and $p(x)$ be a polynomial such that $p(X)$ is a graph. Then $p(X)$ is walk regular.*

Proof Let A be the adjacency matrix of X. From the above theorem the diagonal entries of A^s $\forall s$ are all equal, so as for every element in $\mathcal{A}(X)$. As one of the basis for $\mathcal{A}(X)$ is $\{I, A, A^2, \ldots A^{l-1}\}$ where l is the degree of the minimal polynomial of A. $\square$

From the above result we deduce that if X be a walk regular graph and X respects Y, then Y is also walk regular graph.

Now we will see some symmetrical properties shared by the respectable graphs.

Definition 2 *A graph $X = (V, E)$ is said to be vertex transitive if its automorphism group acts transitively on V. That is for any two vertices $x, y \in V, \exists g \in G$ such that $g(x) = y$.*

Definition 3 *A graph $X = (V, E)$ is said to be* generously transitive *if its automorphism group acts generously transitively on $V(X)$, i.e., if any $x, y \in V$ then $\exists g \in Aut(X)$ such that $g(x) = y$ and $g(y) = x$.*

Every generously transitive graph is transitive. From the Corollary 2.2 we have the following result.

Lemma 2.12 *If X is a generously transitive (or vertex transitive) graph and Y is a polynomial in X, then Y is also a generously transitive (or vertex transitive) graph.*

§3. Algebraic Properties

Let X be a graph with n vertices and A be the adjacency matrix of X. By graph algebra of X, we mean a matrix subalgebra of $M_n(\mathbb{C})$ which contains A. For example $M_n(\mathbb{C})$ and $\mathcal{A}(X)$ are graph algebras of X. If the graph X respects Y, then in this section we will show that the following three graph algebras of X and Y will coincide.

- The *commutant algebra* of a graph Z is the set all matrices over $\mathbb{C}$ which commutes with adjacency matrix of Z.
- The *coherent closure* of a graph Z is the smallest coherent algebra containing the adjacency matrix of Z.

- The *centralizer algebra* of a graph Z is the set all matrices which commute with all automorphisms of Z.

3.1 Coherent Closure of a Graph

Definition 4 *Hadamard product of two $n \times n$ matrices A and B is denoted by $A \odot B$ and is defined as $(A \odot B)_{xy} = A_{xy}B_{xy}$.*

Definition 5 *Two $n \times n$ matrices A and B are said to be* disjoint *if their Hadamard product is the zero matrix.*

Definition 6 *A sub algebra of $M_n(\mathbb{C})$ is called coherent if it contains the matrices I and J and if it is closed under conjugate-transposition and Hadamard multiplication.*

The following result is well known.

Theorem 3.1 *Every coherent algebra contains unique basis of disjoint 0-1 matrices.*

We call the unique basis containing disjoint 0-1 matrices as a standard basis.

Corollary 3.2 *Every $0,1$-matrix in a coherent algebra is sum of one or more matrices in its standard basis.*

Proof Let $\mathcal{M}$ be a coherent algebra over $\mathbb{C}$ with standard basis $\{M_1, \dots M_t\}$. Let $B \in \mathcal{M}$ be a $0,1$-matrix. Then $B = \sum_{i=1}^{t} a_i M_i$ where $a_i \in \mathbb{C}$. $B = B \odot B = \sum_{i=1}^{t} a_i^2 M_i \Rightarrow a_i^2 = a_i$. Hence the result follows. □

Observation 3.3 The intersection of coherent algebras is again a coherent algebra.

Definition 7 *Let $X = (V, E)$ be a graph with adjacency matrix A then any coherent algebra which contains A is called coherent algebra of X.*

Definition 8 *If $X = (V, E)$ be a graph and A is its adjacency matrix then coherent closure of X, denoted by $\langle\langle A \rangle\rangle$ or $\mathcal{CC}(X)$, is the smallest coherent algebra containing A.*

Since $A(X^c) = J - I - A(X)$ consequently $A(X), A(X^c) \in \mathcal{CC}(X) \cap \mathcal{CC}(X^c)$, hence we get the following lemma.

Lemma 3.4 *For every graph X, $\mathcal{CC}(X) = \mathcal{CC}(X^c)$.*

Lemma 3.5 *If the graph X respects Y, then $\mathcal{CC}(X) = \mathcal{CC}(Y)$.*

Proof Since X respects Y, we have $\mathcal{A}(X) = \mathcal{A}(Y) \subseteq \mathcal{CC}(Y)$. Consequently $\mathcal{CC}(Y)$ is a coherent algebra containing $A(X)$ but by definition $\mathcal{CC}(X)$ is the smallest coherent algebra containing $A(X)$. So $\mathcal{CC}(X) \subseteq \mathcal{CC}(Y)$. Similarly we can prove $\mathcal{CC}(Y) \subseteq \mathcal{CC}(X)$. Hence the result follows. □

Clearly, the converse of this result is not true as $\mathcal{CC}(X) = \mathcal{CC}(X^c)$, but X need not respect X^c.

3.2 Centralizer Algebra of a Graph

Definition 9 *Let G be a subset of $n \times n$ permutation matrices forming a group. Then $\mathcal{V}_{\mathbb{C}}(G) = \{A \in M_n(\mathbb{C}) : PA = AP \ \forall P \in G\}$ forms an algebra over $\mathbb{C}$ called the centralizer algebra of the group G.*

Definition 10 *If G is a group acting on a set V, then G also acts on $V \times V$ by $g(x, y) = (g(x), g(y))$. The orbits of G on $V \times V$ are called orbitals. In the context of graphs, the orbitals of graph X are orbitals of its automorphism group $Aut(X)$ acting on the vertex set of X. That is, the orbitals are the orbits of the arcs/non-arcs of the graph $X = (V, E)$. The number of orbitals is called the rank of X.*

An orbital can be represented by a $0-1$ matrix M where M_{ij} is 1 if (i, j) belongs to the orbital. We can associate directed graphs to these matrices. If the matrices are symmetric, then these can be treated as undirected graphs.

Observation 3.6

- The '1' entries of any orbital matrix are either all on the diagonal or all are off diagonal.
- The orbitals containing 1's on the diagonal will be called diagonal orbitals.

Definition 11 *The centralizer algebra of a graph X denoted by $\mathcal{V}(X)$ is the centralizer algebra of its automorphism group.*

Theorem 3.7([4]) *$\mathcal{V}_{\mathbb{C}}(G)$ is a coherent algebra and orbitals of* Aut$X$ *acting on the vertex set of X form its unique 0-1 matrix basis.*

$\mathcal{V}(X) = \mathcal{V}(X^c)$ follows from the fact that $\mathrm{Aut}(X) = \mathrm{Aut}(X^c)$. $\mathcal{CC}(X)$ is the smallest coherent algebra of X and $\mathcal{V}(X)$ is a coherent algebra of X so $\mathcal{CC}(X) \subseteq \mathcal{V}(X)$. So for any graph X we have $\mathcal{A}(X) \subseteq \mathcal{CC}(X) \subseteq \mathcal{V}(X)$. The following result follows from the Corollary 2.3.

Lemma 3.8 *If the graph X respects the graph Y, then $\mathcal{V}(X) = \mathcal{V}(Y)$.*

Now we will see a consequence of above result. For that we need the following definition.

Definition 12(Robert A.Beezer [1]) *A graph $X = (V, E)$ is* orbit polynomial *graph if each orbital matrix is member of $\mathcal{A}(X)$. That is each orbital matrix is a polynomial in A.*

Lemma 3.9 *X is an orbit polynomial graph if and only if $\mathcal{A}(X) = \mathcal{V}(X)$.*

If X is an orbit polynomial graph, then we have $\mathcal{A}(X) = \mathcal{CC}(X) = \mathcal{V}(X)$.

Corollary 3.10 *Let X be an orbit polynomial graph and suppose X respects the graph Y, then Y is also an orbit polynomial graph.*

Corollary 3.11 *If X is an orbit polynomial graph and X^c is connected then X^c is orbit polynomial graph.*

If X is an orbit polynomial graph and X^c is connected, then we have $\mathcal{A}(X) = \mathcal{A}(X^c) = \mathcal{CC}(X) = \mathcal{CC}(X^c) = \mathcal{V}(X) = V(X^c)$.

3.3 Commutant algebra of a graph

The commutant algebra of graph X, denoted by $C[X]$ is the set of all matrices which commutes with A. It is shown in (Davis [5]) that dim($C[A]$)=sum of the squares of the multiplicities of eigenvalues of A. Hence the following lemma.

Lemma 3.12 $\mathcal{A}(X) = C[X]$ *if and only if all eigenvalues of X are distinct.*

Lemma 3.13 *If the graph X respects the graph Y then $C[X] = C[Y]$.*

Proof Notice that

$$B \in C[X] \Leftrightarrow BA(X) = A(X)B \Leftrightarrow BA(Y) = A(Y)B \Leftrightarrow B \in C[Y].$$

We get the result. □

§4 Polynomial Equivalence

Let $\mathcal{G}_n$ be the set of all graphs with n vertices. We define a relation $\mathcal{R}$ on $\mathcal{G}_n$ as $X\mathcal{R}Y \Leftrightarrow X$ respects Y. It is easy to see that $\mathcal{R}$ is an equivalence relation on $\mathcal{G}_n$. Now for a given graph X, our objective is to find the equivalence class $[X]$ under the equivalence relation $\mathcal{R}$. First we identify a set $[X]$ with a set in polynomial algebra $\mathbb{C}[x]$. For that we need the following notations and definitions.

$\mathbb{C}[A]$ denote the set of all matrices which are polynomials in the square matrix A. It is easy to see that $\mathbb{C}[A] \cong \mathbb{C}[x]/\langle p(x)\rangle$ where $\langle p(x)\rangle$ is the ideal in $\mathbb{C}[x]$ generated by $p(x)$, which is the minimal polynomial of A. Consequently if $B \in \mathbb{C}[A]$, then there exists a unique polynomial $f_B(x)$ called representor polynomial of B such that $\deg(f_B(X)) \le \deg(p(x))$ and $f_B(A) = B$.

Definition 13 *Let A be a square matrix and $f(x)$ be a polynomial. We say that $f(x)$ respects spec(A) if $f(\lambda_i) \neq f(\lambda_j)$ for λ_i and λ_j distinct eigenvalues of A.*

The following result is given by Paul M.Weichsel [7].

Lemma 4.1 *Let A be diagonalizable matrix over a field and $f(x) \in \mathbb{C}[x]$. Then $f(x)$ respects spec(A) if and only if there exists a polynomial $g(x) \in \mathbb{C}[x]$ such that $g(f(A)) = A$.*

Proof Let $B = f(A)$. Clearly $\mathbb{C}[B] \subseteq \mathbb{C}[A]$. Since A is diagonalizable, so is B. Consequently A is a polynomial in B if and only if $\mathbb{C}[B] = \mathbb{C}[A]$ if and only if $|spec(A)| = |spec(B)|$ if and only if $f(x)$ respects $spec(A)$. □

Now let A be the adjacency matrix of a graph X and we denote

$$F_X = \{f(x) \in \mathbb{C}[x] | \deg(f(x)) \leq \deg(p(x)) \text{ and } f(A) \text{ is a 0,1-matrix}\},$$
$$H_X = \{g(x) \in F_X | g(x) \text{ respects } spec(A)\}.$$

Now one can easily verify that finding the set $[X]$ is equivalent to finding the set H_X. By definition, in order to find H_X we need to find F_X but for a given graph X finding F_X seems difficult. Let X be a graph with the property $\mathcal{A}(X) = \mathcal{CC}(X)$, then from Corollary 3.2 it is easy to evaluate F_X. Distance regular graphs and orbit polynomial graphs satisfy $\mathcal{A}(X) = \mathcal{CC}(X)$ for details one can refer Robert A.Beezer [2] and Paul M.Weichsel [7].

The following theorem shows that if X is a connected vertex transitive graph with a prime number of vertices then X respects Y if and only if $\mathrm{Aut}(X) = \mathrm{Aut}(Y)$.

Theorem 4.2(Robert A.Beezer [2]) *Suppose that X is a connected, vertex transitive graph with a prime number of vertices. Let $p(x)$ be a polynomial such that $p(X)$ is a connected graph, and $Aut(X) = Aut(p(X))$. Then $p(x)$ respects $spec((A(X))$.*

Comments In spite of these results, there are many properties which are not shared by the respectable graphs. We illustrate few of them with examples. Let C_n denote the cycle graph with n vertices, then C_n respects C_n^c for $n \geq 5$. It is known that C_{2n} is bipartite for every n, but C_{2n}^c is not bipartite for $n \geq 3$. For $n \geq 3$, C_{2n} is Eulerain graph but C_{2n}^c is not. C_n is planar graph $\forall n$ where as C_n^c is not planar for $n \geq 9$ as every finite, simple, planar graph has a vertex of degree less than 6. Petersen graph is not Hamiltonian graph but from Dirac's theorem its compliment (respects Petersen graph) is a Hamiltonian graph.

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Common Fixed Points for Pairs of Weakly Compatible Mappings

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Abstract: In this note we establish a common fixed point theorem for a quadruple of self mappings satisfying a common (E.A) property on a metric space satisfying weakly compatibility and a generalized Φ- contraction. Our results improve and extend some known results.

Key Words: Common fixed points, weakly compatible mappings, generalized Φ- contraction, a common (E.A) property, Smarandache metric multi-space.

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§1. Introduction

For an integer $n \geq 1$, a Smarandache metric multi-space $\widetilde{S}$ is a union $\bigcup\limits_{i=1}^{n} A_i$ of spaces $A_1, A_2, \cdots,$ A_n, distinct two by two with metrics $\rho_1, \rho_2, \cdots, \rho_n$ such that (A_i, ρ_i) is a metric space for integers $1 \leq i \leq n$. In 1986, the notion of compatible mappings which generalized commuting mappings, was introduced by Jungck [3]. This has proven useful for generalization of results in metric fixed point theory for single-valued as well as multi-valued mappings. Further in 1998, the more general class of mappings called weakly compatible mappings was introduced by Jungck and Rhoades [4]. Recall that self mappings S and T of a metric space (X, d) are called weakly compatible if $Sx = Tx$ for some $x \in X$ implies that $STx = TSx$.

Recently Aamri et al. [1] introduced the following notion for a pair of maps as:

Definition 1.1 *Let S and T be two self mappings of a metric space (X, d). S and T are said to satisfy the property (E.A), if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.*

Most recently, Y. Liu et al. [5] defined a common property (E.A) for pairs of mappings as

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follows:

Definition 1.2 *Let $A, B, S, T : X \to X$. The pairs (A, S) and (B, T) satisfy a common property (E.A) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t \in X.$$

If $B = A$ and $S = T$ in above, we obtain the definition of property (E.A).

Example 1.3 Let A, B, S and T be self maps on $X = [0, 1]$, with the usual metric $d(x, y) = |x - y|$, defined by:

$$Ax = \begin{cases} 1 - \frac{x}{2} & when \quad x \in [0, \frac{1}{2}), \\ 1 & when \quad x \in [\frac{1}{2}, 1]). \end{cases}$$

$$Sx = \begin{cases} 1 - 2x & when \quad x \in [0, \frac{1}{2}), \\ 1 & when \quad x \in [\frac{1}{2}, 1]). \end{cases}$$

$Bx = 1 - x$ and $Tx = 1 - \dfrac{x}{3}$, $\forall\, x \in X$. Let $\{x_n\}$ and $\{y_n\}$ be a sequences defined by $x_n = \dfrac{1}{n+1}$ and $y_n = \dfrac{1}{n^2+1}$, then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1 \in X$. Thus a common (E.A) property is satisfied.

In this paper we prove some common fixed point theorems for a quadruple of weak compatible self mappings of a metric space satisfying a common (E.A) property, a special Smarandache metric multi-space $\bigcup\limits_{i=1}^{n}(A_i, \rho_i)$ for $n = 1$ and a generalized Φ-contraction. These theorems extend and generalize results of Pathak et al. [6] and [7].

§2. Preliminaries

Now onwards, we denote by Φ the collection of all functions $\varphi : [0, \infty) \to [0, \infty)$ which are upper semi-continuous from the right, non-decreasing and satisfy $lim_{s\to t+}$ sup $\varphi(s) < t$, $\varphi(t) < t$ for all $t > 0$.

Let X denote a metric space endowed with metric d and let $\mathbb{N}$ denote the set of natural numbers.

Now, let A, B, S and T be self-mappings of X such that

$$\begin{aligned}
&[d^p(Ax, By) + a\ d^p(Sx, Ty)]d^p(Ax, By) \\
&\leq a\ max\{d^p(Ax, Sx)d^p(By, Ty), d^q(Ax, Ty)d^{q'}(By, Sx)\} \\
&+max\{\varphi_1(d^{2p}(Sx, Ty)), \varphi_2(d^r(Ax, Sx)d^{r'}(By, Ty)), \\
&\varphi_3(d^s(Ax, Ty)d^{s'}(By, Sx)), \\
&\varphi_4(\frac{1}{2}[d^l(Ax, Ty)d^{l'}(Ax, Sx) + d^l(By, Sx))d^{l'}(By, Ty)\} \qquad (2.1)
\end{aligned}$$

for all $x, y \in X, \varphi_i \in \Phi(i = 1, 2, 3, 4), a, p, q, q', r, r', s, s', l, l' \geq 0$ and $2p = q + q' = r + r' = s + s' = l + l'$. The condition (2.1) is commonly called a generalized Φ-contraction.

§3. Main Results

The following theorems are our main results in this section.

Theorem 3.1 *Let A, B, S and T be self mappings of a metric space (X, d) satisfying (2.1).If the pairs (A, S) and (B, T) satisfy a common (E.A) property, are weakly compatible and that $T(X)$ and $S(X)$ are closed subsets of X, then A, B, S and T have a unique common fixed point in X.*

Proof. Since (A, S) and (B, T) satisfy a common property (E.A). Then there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$$

for some $z \in X$. Assume that $S(X)$ and $T(X)$ are closed subspaces of X. Then, $z = Su = Tv$ for some $u, v \in X$. Then by using (2.1) with $x = x_n$ and $y = v$, we have

$$\begin{aligned}[d^p(Ax_n, Bv) + a\ d^p(Sx_n, Tv)]d^p(Ax_n, Bv) \leq a \max\{&d^p(Ax_n, Sx_n)d^p(Bv, Tv),\\ &d^q(Ax_n, Tv)d^{q'}(Bv, Sx_n)\} + max\{\varphi_1(d^{2p}(Sx_n, Tv)),\\ &\varphi_2(d^r(Ax_n, Sx_n)d^{r'}(Bv, Tv)), \varphi_3(d^s(Ax_n, Tv)d^{s'}(Bv, Sx_n)),\\ &\varphi_4(\frac{1}{2}[d^l(Ax_n, Tv)d^{l'}(Ax_n, Sx_n) + d^l(Bv, Sx_n))d^{l'}(Bv, Tv)]),\end{aligned}$$

taking $\lim_{n\to\infty}$, we obtain

$$\begin{aligned}[d^p(z, Bv) + a\ d^p(z, Tv)]d^p(z, Bv) \leq a \max\{&d^p(z, z)d^p(Bv, z), d^q(z, Tv)d^{q'}(Bv, z)\}\\ &+ max\{\varphi_1(d^{2p}(z, Tv)), \varphi_2(d^r(z, z)d^{r'}(Bv, z)),\\ &\varphi_3(d^s(z, Tv)d^{s'}(Bv, z)), \varphi_4(\frac{1}{2}[d^l(z, Tv)d^{l'}(z, z)\\ &+ d^l(Bv, z))d^{l'}(Bv, z)])\},\end{aligned}$$

or $$d^{2p}(z, Bv) \leq max\{\varphi_1(0), \varphi_2(0), \varphi_3(0), \varphi_4(\tfrac{1}{2}d^{l+l'}(Bv, z))\},$$
or $$\begin{aligned}d^{2p}(z, Bv) \leq max\{&\varphi_1(d^{2p}(z, Bv)), \varphi_2(d^{r+r'}(z, Bv),\\ &\varphi_3(d^{s+s'}(z, Bv)), \varphi_4(\tfrac{1}{2}d^{l+l'}(Bv, z))\}.\end{aligned}$$

This together with a well known result of Chang [2] which states that if $\varphi_i \in \Phi$ where $i \in I$ (some indexing set), then there exists a $\varphi \in \Phi$ such that max $\{\varphi_i, i \in I\} \leq \varphi(t)$ for all $t > 0$; imply

$$d^{2p}(z, Bv) \leq \varphi(d^{2p}(z, Bv)) \ < d^{2p}(z, Bv),$$

a contradiction. This implies that $z = Bv$. Therefore $Tv = z = Bv$. Hence it follows by the weak compatibility of the pair (B, T) that $BTv = TBv$, that is $Bz = Tz$.

Now, we shall show that z is a common fixed point of B and T. For this put $x = x_n$ and $y = z$ in (2.1), we have

$$[d^p(Ax_n, Bz) + a\ d^p(Sx_n, Tz)]d^p(Ax_n, Bz) \le a\ \max\{d^p(Ax_n, Sx_n)d^p(Bz, Tz),$$
$$d^q(Ax_n, Tz)d^{q'}(Bz, Sx_n)\} + max\{\varphi_1(d^{2p}(Sx_n, Tz)),$$
$$\varphi_2(d^r(Ax_n, Sx_n)d^{r'}(Bz, Tz)), \varphi_3(d^s(Ax_n, Tz)d^{s'}(Bz, Sx_n)),$$
$$\varphi_4(\frac{1}{2}[d^l(Ax_n, Tz)d^{l'}(Ax_n, Sx_n) + d^l(Bz, Sx_n))d^{l'}(Bz, Tz)\}.$$

Letting $n \to \infty$ with the help of the fact that $lim_{n\to\infty}Ax_n = z = lim_{n\to\infty}Sx_n$ and $Bz = Tz$, we get

$$[d^p(z, Bz) + ad^p(z, Tz)]d^p(z, Bz) \le a\ max\{d^p(z, z)d^p(Bz, z), d^q(z, Tz)d^{q'}(Bz, z)\}$$
$$+ max\{\varphi_1(d^{2p}(z, Tz)), \varphi_2(d^r(z, z)d^{r'}(Bz, z)), \varphi_3(d^s(z, Tz)d^{s'}(Bz, z)),$$
$$\varphi_4(\frac{1}{2}[d^l(z, Tz)d^{l'}(z, z) + d^l(Bz, z))d^{l'}(Bz, z)])\},$$

or
$$d^{2p}(z, Bz) + a\ d^{2p}(z, Bz) \le a\ d^{q+q'}(Bz, z) + max\{\varphi_1(d^{2p}(z, Bz)), \varphi_2(0), \varphi_3(d^{s+s'}(z, Bz)), \varphi_4(0)\},$$
or
$$(1 + a)d^{2p}(z, Bz) \le a\ d^{q+q'}(Bz, z)\} + max\{\varphi_1(d^{2p}(z, Bz)), \varphi_2(0), \varphi_3(d^{s+s'}(z, Bz)), \varphi_4(0)\},$$
or
$$d^{2p}(z, Bz) \le \tfrac{a}{1+a}d^{q+q'}(Bz, z) + \tfrac{1}{1+a}max\{\varphi_1(d^{2p}(z, Bz)), \varphi_2(0), \varphi_3(d^{s+s'}(z, Bz)), \varphi_4(0)\} < d^{2p}(z, Bz),$$

a contradiction. So $z = Bz = Tz$. Thus z is a common fixed point of B and T.

Similarly we can prove that z is a common fixed point of A and S. Thus z is the common fixed point of A, B, S and T. The uniqueness of z as a common fixed point of A, B, S and T can easily be verified. □

Remark 3.3 Our Theorem 3.1 extends theorem 2.1 of Pathak et al. [6].

In Theorem 3.1, if we put $a = 0$ and $\varphi_i(t) = ht$ $(i = 1, 2, 3, 4)$, where $0 < h < 1$, we get the following corollary:

Corollary 3.4 *Let A, B, S and T be self mappings of a metric space X. If the pairs (A, S) and (B, T) satisfy a common (E.A) property and*

$$d^{2p}(Ax, By) \le h\ max\{d^{2p}(Sx, Ty), d^r(Ax, Sx)d^{r'}(By, Ty), d^s(Ax, Ty)$$
$$d^{s'}(By, Sx)), \frac{1}{2}[d^l(Ax, Ty)d^{l'}(Ax, Sx) + d^l(By, Sx))d^{l'}(By, Ty)\} \quad (2.2)$$

for all $x, y \in X, \varphi_i \in \Phi$ (i=1,2,3,4), a, p, q, $q', r, r', s, s', l, l' \ge 0$ and $2p = q + q' = r + r' = s + s' = l + l'$. If the pairs (A, S) and (B, T) are weakly compatible and that $T(X)$ and $S(X)$ are closed, then A, B, S and T have a unique common fixed point in X.

Especially when

$$\max\{d^{2p}(Sx,Ty), d^r(Ax,Sx)d^{r'}(By,Ty), d^s(Ax,Ty)d^{s'}(By,Sx)),$$
$$\frac{1}{2}[d^l(Ax,Ty)d^{l'}(Ax,Sx)+d^l(By,Sx))d^{l'}(By,Ty)\} = d^{2p}(Sx,Ty),$$

it generalizes Corollary 3.9 of Pathak et al. [7].

In Theorem 3.1, if we take $S = T = I_X$ (the identity mapping on X), then we have the following corollary:

Corollary 3.5 *Let A and B be self mappings of a complete metric space X satisfying the following condition:*

$$[d^p(Ax,By)+a\ d^p(x,y)]d^p(Ax,By) \le a\ max\{d^p(Ax,x)d^p(By,y),$$
$$d^q(Ax,y)d^{q'}(By,x)\} + max\{\varphi_1(d^{2p}(x,y)), \varphi_2(d^r(Ax,x)d^{r'}(By,y)),$$
$$\varphi_3(d^s(Ax,y)d^{s'}(By,x)), \varphi_4(\tfrac{1}{2}[d^l(Ax,y)d^{l'}(Ax,x)+d^l(By,x))d^{l'}(By,y)\}$$

for all $x,y \in X, \varphi_i \in \Phi$ $(i=1,2,3,4), a,p,q,q',r,r',s,s',l,l' \ge 0$ and $2p = q+q' = r+r' = s+s' = l+l'$, then A and B have a unique common fixed point in X.

As an immediate consequences of Theorem 3.1 with $S = T$, we have the following:

Corollary 3.6 *Let A, B, and S be self-mappings of X such that (A,S) and (B,S) satisfy a common (E.A) property and*

$$d^{2p}(Ax,By) \le a\ max\{d^p(Ax,Sx)d^p(By,Sy), d^q(Ax,Sy)d^{q'}(By,Sx)\}$$
$$+ max\{\varphi_2(d^r(Ax,Sx)d^{r'}(By,Sy)), \varphi_3(d^s(Ax,Sy)d^{s'}(By,Sx)),$$
$$\varphi_4(\tfrac{1}{2}[d^l(Ax,Sy)d^{l'}(Ax,Sx)+d^l(By,Sx))d^{l'}(By,Sy)\} \tag{2.3}$$

for all $x,y \in X, \varphi_i \in \Phi$ $(i=1,2,3,4), a,p,q,q',r,r',s,s',l,l' \ge 0$ and $2p = q+q' = r+r' = s+s' = l+l'$. If the pairs (A,S) and (B,S) are weakly compatible and that $S(X)$ is closed, then A, B and S have a unique common fixed point in X.

Theorem 3.7 *Let S, T and A_n $(n \in \mathbb{N})$ be self mappings of a metric space (X,d). Suppose further that the pairs (A_{2n-1},S) and (A_{2n},T) are weakly compatible for any $n \in \mathbb{N}$ and satisfying a common (E.A) property. If $S(X)$ and $T(X)$ are closed and that for any $i \in N$, the following condition is satisfied for all $x,y \in X$*

$$[d^p(A_ix,A_{i+1}y)+a\ d^p(Sx,Ty)]d^p(A_ix,A_{i+1}y)$$
$$\le a\max\{d^p(A_ix,Sx)d^p(A_{i+1}y,Ty),$$
$$d^q(A_ix,Ty)d^{q'}(A_{i+1}y,Sx)\} + max\{\varphi_1(d^{2p}(Sx,Ty)),$$
$$\varphi_2(d^r(A_ix,Sx)d^{r'}(A_{i+1}y,Ty)), \varphi_3(d^s(A_ix,Ty)d^{s'}(A_{i+1}y,Sx)),$$
$$\varphi_4(\frac{1}{2}[d^l(A_ix,Ty)d^{l'}(A_ix,Sx)+d^l(A_{i+1}y,Sx))d^{l'}(A_{i+1}y,Ty)\}$$

where $\varphi_i \in \Phi(i=1,2,3,4)$, $a,p,q,q',r,r',s,s',l,l, \ge 0$ and $2p = q+q' = r+r' = s+s' = l+l'$, then S, T and $A_n(n \in \mathbb{N})$ have a common fixed point in X.

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Some Results on
Generalized Multi Poly-Bernoulli and Euler Polynomials

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Abstract: The Arakawa-Kaneko zeta function has been introduced ten years ago by T. Arakawa and M. Kaneko in [22]. In [22], Arakawa and Kaneko have expressed the special values of this function at negative integers with the help of generalized Bernoulli numbers $B^{(k)}$ called *poly-Bernoulli numbers*. Kim-Kim [4] introduced Multi poly- Bernoulli numbers and proved that special values of certain zeta functions at non-positive integers can be described in terms of these numbers. The study of Multi poly-Bernoulli and Euler numbers and their combinatorial relations has received much attention [2,4,6,7,12,13,14,19,22,27]. In this paper we introduce the generalization of Multi poly-Bernoulli and Euler numbers and consider some combinatorial relationships of the Generalized Multi poly-Bernoulli and Euler numbers of higher order. The present paper deals with Generalization of Multi poly-Bernouli numbers and polynomials of higher order. In 2002, Q. M. Luo and et al (see [11, 23, 24]) defined the generalization of Bernoulli polynomials and Euler numbers. Some earlier results of Luo in terms of generalized Multi poly-Bernoulli and Euler numbers, can be deduced. Also we investigate some relationships between Multi poly-Bernoulli and Euler polynomials.

Key Words: Generalized Multi poly-Bernoulli polynomials, generalized Multi poly-Euler polynomials, stirling numbers, polylogarithm, Multi- polylogarithm.

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§1. Introduction

Bernoulli numbers are the signs of a very strong bond between elementary number theory, complex analytic number theory, homotopy theory(the J-homomorphism, and stable homotopy groups of spheres), differential topology(differential structures on spheres), the theory of modular forms(Eisenstein series) and p-adic analytic number theory(the p-adic L-function) of

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Mathematics. For $n \in Z, n \geq 0$, Bernulli numbers B_n originally arise in the study of finite sums of a given power of consecutive integers. They are given by $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, ...$, with $B_{2n+1} = 0$ for $n > 1$, and

$$B_n = -\frac{1}{n+1} \sum_{m=0}^{n-1} \binom{n+1}{m} B_m, \quad n \geq 1 \tag{1}$$

The modern definition of Bernoulli numbers B_n can be defined by the contour integral

$$B_n = \frac{n!}{2\pi i} \oint \frac{z}{e^z - 1} \frac{dz}{z^{n+1}}, \tag{2}$$

where the contour encloses the origin, has radius less than 2π.

Also Bernoulli polynomials $B_n(x)$ are usualy defined(see[1], [4], [5])by the generating function

$$G(x,t) = \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi \tag{3}$$

and consequently, Bernoulli numbers $B_n(0) := B_n$ can be obtained by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

Bernoulli polynomials, first studied by Euler (see[1]), are employed in the integral representation of differentiable periodic functions, and play an important role in the approximation of such functions by means of polynomials (see[14]-[18]).

Euler polynomials $E_n(x)$ are defined by the generating function

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad |t| < \pi \tag{4}$$

Euler numbers E_n can be obtained by the generating function

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \tag{5}$$

The first four such polynomials, are

$$B_0(x) = 1, B_1(x) = x - 1/2, B_2(x) = x^2 - x + 1/6$$
$$B_3(x) = x^3 - 3/2x^2 + 1/2x, ...$$

and

$$E_0(x) = 1, E_1(x) = x - 1/2, E_2(x) = x^2 - x,$$
$$E_3(x) = x^3 - 3/2x^2 + 1/4, ...$$

Euler polynomials are strictly connected with Bernoulli ones, and are used in the Taylor expansion in a neighborhood of the origin of trigonometric and hyperbolic secant functions.

In the sequel, we list some properties of Bernoulli and Euler numbers and polynomials as well as recurrence relations and identities.

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \tag{6}$$

$$E_n(x) = \frac{1}{n+1} \sum_{k=1}^{n+1} (2 - 2^{k+1}) \binom{n+1}{k} B_k x^{n+1-k}. \tag{7}$$

$$B_n(x+1) - B_n(x) = nx^{n-1}, \tag{8}$$

$$E_n(x+1) + E_n(x) = 2x^n. \tag{9}$$

Lemma 1.1(see[20],[21]) *For any integer $n \geq 0$, we have*

$$B_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) \tag{10}$$

$$E_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) \tag{11}$$

Consequently, from (8), (9) and lemma 1.1, we obtain,

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k(x) = (n+1)x^n \tag{12}$$

$$\sum_{k=0}^{n} \binom{n}{k} E_k(x) + E_n(x) = 2x^n. \tag{13}$$

Lemma 1.3 *For any positive integer $n \geq 0$, we have*

$$B_n(px) = p^{n-1} \sum_{r=0}^{p-1} B_n(x + \frac{r}{p}) \ (p \textit{ is a positive integer}) \tag{14}$$

$$E_n(px) = p^n \sum_{r=0}^{p-1} (-1)^r E_n(x + \frac{r}{p}) \ (p \textit{ is an odd integer}) \tag{15}$$

Let us briefly recall $k-th$ polylogarithm. The polylogarithm is a special function $Li_k(z)$, that is defined by the sum

$$Li_k(z) := \sum_{s=1}^{\infty} \frac{z^s}{s^k} \tag{16}$$

For formal power series $Li_k(z)$ is the $k-th$ polylogarithm if $k \geq 1$, and a rational function if $k \leq 0$. The name of the function come from the fact that it may alternatively be defined as the repeated integral of itself, namely that

$$Li_{k+1}(z) = \int_0^z \frac{Li_k(t)}{t} dt \tag{17}$$

for integer values of k, we have the following explicit expressions

$$Li_1(z) = -\log(1-z),\ Li_0(z) = \frac{z}{1-z}\ Li_{-1}(z) = \frac{z}{(1-z)^2}$$
$$Li_{-2}(z) = \frac{z(1+z)}{(1-z)^3},\ Li_{-3}(z) = \frac{z(1+4z+z^2)}{(1-z)^4}, ...$$

The integral of the Bose-Einstein distribution is expressed in terms of a polylogarithm,

$$Li_{k+1}(z) = \frac{1}{\Gamma(k+1)} \int_0^\infty \frac{t^k}{\frac{e^t}{z} - 1} dt \tag{18}$$

Lemma 1.3(see[18]) *For $n \in N \cup \{0\}$, we have an explicit formula for $Li_{-n}(z)$ as follow*

$$Li_{-n}(z) = \sum_{k=1}^{n+1} \frac{(-1)^{n+k+1}(k-1)!S(n+1,k)}{(1-z)^k} \tag{19}$$
$$(n = 1, 2, ...)$$

where $s(n,k)$ are Stirling numbers of the second kind.

Now, we introduce the generalization of $Li_k(z)$. Let r be an integer with a value greater than one.

Definition 1.1 *Let $k_1, k_2, ...k_r$ be integers. The generalization of polylogarithm are defined by*

$$Li_{k_1,k_2,...,k_r}(z) = \sum_{\substack{m_1,m_2,...,m_r \in \mathcal{Z} \\ 0<m_1<m_2<...<m_r}} \frac{z^{m_r}}{m_1^{k_1}...m_r^{k_r}} \tag{20}$$

The rational numbers $B_n^{(k)}, (n = 0, 1, 2, ...)$ are said to be poly-Bernoulli numbers if they satisfy

$$\frac{Li_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!} \tag{21}$$

In addition, for any $n \geq 0$, $B_n^{(1)}$ is the classical Bernoulli number, B_n(see[7], [12]). Also , the rational numbers $H_n^{(k)}(u), (n = 0, 1, 2, ...)$are said to be poly-Euler numbers if they satisfy

$$\frac{Li_k(1-e^{(1-u)})}{u-e^t} = \sum_{n=0}^{\infty} H_n^{(k)}(u) \frac{t^n}{n!} \tag{22}$$

where u is an algebraic real number and $k \geq 1$.(see[13],[19])

Let us now introduce a generalization of poly-Bernoulli numbers, making use of $Li_{k_1,...,k_r}(z)$.

Definition 1.1(see[7]) *Multi poly-Bernoulli numbers $B_n^{(k_0,...,k_r)}, (n = 0, 1, 2, ...)$ are defined for each integer $k_1, k_2, ..., k_r$ by the generating series*

$$\frac{Li_{(k_1,k_2,...,k_r)}(1-e^{-t})}{(1-e^{-t})^r} = \sum_{n=0}^{\infty} B_n^{(k_1,...,k_r)} \frac{t^n}{n!} \tag{23}$$

By Definition 1.2, the left hand side of (23) is

$$\frac{1}{1^{k_1}2^{k_2}...r^{k_r}} + \sum_{\substack{0<m_1<...<m_r \\ m_r \neq r}} \frac{(1-e^{-t})^{m_r-r}}{m_1^{k_1}...m_r^{k_r}} \tag{24}$$

hence we have

$$B_0^{(k_1,..,k_r)} = \frac{1}{1^{k_1}2^{k_2}...r^{k_r}} \tag{25}$$

$$B_1^{(k_1,...k_r)} = \sum_{0<m_1<...<m_r} \frac{1}{m_1^{k_1}...m_{r-1}^{k_{r-1}}(r+1)^{k_r}} \tag{26}$$

Definition 1.3 *Multi poly-Euler numbers* $H_n^{(k_1,...,k_r)}, (n = 0, 1, ...)$ *are defined for each integer* $k_1, ..., k_r$ *by the generating series*

$$\frac{Li_{(k_1,...,k_r)}(1-e^{(1-u)})}{(u-e^t)^r} = \sum_{n=0}^{\infty} H_n^{(k_1,...,k_r)}(u)\frac{t^n}{n!} \tag{27}$$

Kaneko [6] presented the following recurrence formulae for poly-Bernoulli numbers which we state hear.

Theorem 1.1(Kaneko)([2,6,14,22]) *For any* $k \in Z$ *and* $n \geq 0$,*we have*

$$B_n^{(k)} = \frac{1}{n+1}\left\{B_n^{(k-1)} - \sum_{m=1}^{n-1}\binom{n}{m-1}B_m^{(k)}\right\} \tag{28}$$

$$B_n^{(k)} = (-1)^n \sum_{k=1}^{n+1} \frac{(-1)^{m-1}(m-1)!\left\{\begin{matrix} n \\ m-1 \end{matrix}\right\}}{m^k} \tag{29}$$

$$B_n^{(-k)} = \sum_{j=0}^{\min(n,k)} (j!)^2 \left\{\begin{matrix} n+1 \\ j+1 \end{matrix}\right\}\left\{\begin{matrix} k+1 \\ j+1 \end{matrix}\right\} \quad (n, k \geq 0) \tag{30}$$

$$B_n^{(-k)} = B_k^{(-n)} \quad (n, k \geq 0) \tag{31}$$

$$B_n^{(k)} = \sum_{m=0}^{n}(-1)^m\binom{n}{m}B_{n-m}^{(k-1)}\left\{\sum_{l=0}^{m}\frac{(-1)^l}{n-l+1}\binom{m}{l}B_l^{(1)}\right\} \tag{32}$$

where

$$\left\{\begin{matrix} n \\ m \end{matrix}\right\} = \frac{(-1)^m}{m!}\sum_{l=0}^{m}(-1)^l\binom{m}{l}l^n \quad n, m \geq 0 \tag{33}$$

called the second type stirling numbers.

Y.Hamahata and H.Masubuchi in [12], presented the following recurrence formulae for Multi poly-Bernoulli numbers.

Theorem 1.2(H.Masubuchi & Y.Hamahata) *For $n \geq 0$ and $(k_1, ..., k_r \in Z)$ we have*

$$B_n^{(k_1,\ldots,k_r)} = (-1)^n \sum_{m_r=r}^{n+r} \left(\sum_{0<m_1<\ldots<m_r} \frac{(-1)^{m_r-r}(m_r-r)! \begin{Bmatrix} n \\ m_r-r \end{Bmatrix}}{m_1^{k_1}...m_r^{k_r}} \right) \tag{34}$$

If $k_r \neq 1$and $n \geq 1$, then

$$B_n^{(k_1,\ldots,k_r)} = \frac{1}{n+r} \left\{ B_n^{(k_1,\ldots,k_{r-1},k_r-1)} - \sum_{m=1}^{n-1} \binom{n}{m-1} B_m^{(k_1,\ldots,k_r)} \right\} \tag{35}$$

If $k_r = 1$ and $n \geq 1$,then

$$B_n^{(k_1,\ldots,k_{r-1},1)} = \frac{1}{n+r} \left\{ B_n^{(k_1,\ldots,k_r-1)} - \sum_{m=0}^{n-1} (-1)^{n-m} \left\{ r\binom{n}{m} + \binom{n}{m-1} \right\} B_m^{(k_1,\ldots,k_{r-1},1)} \right\} \tag{36}$$

Also, they proved (see[1]) if

$$B[r]_n^{(k)} = B_n^{(\overbrace{0,\ldots,0}^{r-1},k)} \tag{37}$$

then for $n, k \geq 0$, we have

$$B[r]_n^{(-k)} = B[r]_k^{(-n)} \tag{38}$$

In [23], [24], Q.M.Luo, F.Oi and L.Debnath defined the generalization of Bernoulli and Euler polynomials $B_n(x, a, b, c)$ and $E_n(x, a, b, c)$ respectively, which are expressed as follows

$$\frac{t}{b^t - a^t} c^{xt} = \sum_{k=0}^{\infty} B_k(x, a, b, c) \frac{t^k}{k!} \tag{39}$$

$$\frac{2c^{xt}}{b^t + a^t} = \sum_{k=0}^{\infty} E_k(x, a, b, c) \frac{t^k}{k!} \tag{40}$$

In this paper, by the method of Q.M.Luo and et al [11], we give some properties on generalized Multi poly-Bernoulli and Euler polynomials

Definition 1.4 *Let $a, b > 0$ and $a \neq b$. The generalized Multi poly-Bernoulli numbers $B_n^{(k_1,\cdots,k_r)}(a, b)$, the generalized Multi poly-Bernoulli polynomials*

$$B_n^{(k_1,\cdots,k_r)}(x, a, b) \quad and \quad B_n^{(k_1,\cdots,k_r)}(x, a, b, c)$$

are defined by the following generating functions, respectively;

$$\frac{Li_{(k_1,\ldots,k_r)}(1-(ab)^{-t})}{(b^t-a^{-t})^r}=\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r)}(a,b)\frac{t^n}{n!},\quad |t|<\frac{2\pi}{|\ln a+\ln b|} \tag{41}$$

$$\frac{Li_{(k_1,\ldots,k_r)}(1-(ab)^{-t})}{(b^t-a^{-t})^r}e^{rxt}=\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r)}(x;a,b)\frac{t^n}{n!},\quad |t|<\frac{2\pi}{|\ln a+\ln b|} \tag{42}$$

$$\frac{Li_{(k_1,\ldots,k_r)}(1-(ab)^{-t})}{(b^t-a^{-t})^r}c^{rxt}=\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r)}(x;a,b,c)\frac{t^n}{n!},\quad |t|<\frac{2\pi}{|\ln a+\ln b|} \tag{43}$$

Definition 1.5 *Let $a,b>0$, and $a\neq b$, the generalized Multi poly-Euler numbers $H_n^{(k_1,\ldots,k_r)}(u;a,b)$,the generalized multi poly-Euler polynomial $H_n^{k_1,\ldots,k_r}(x;u,a,b)$ and $H_n^{k_1,\ldots,k_r}(x;u,a,b,c)$ are defined by the following generating functions, respectively,*

$$\frac{Li_{(k_1,\ldots,k_r)}(1-e^{(1-u)})}{(ua^{-t}-b^t)^r}=\sum_{n=0}^{\infty}H_n^{(k_1,\ldots,k_r)}(u,a,b)\frac{t^n}{n!},\quad |t|<\frac{2\pi}{|\ln a+\ln b|} \tag{44}$$

$$\frac{Li_{(k_1,\ldots,k_r)}(1-e^{(1-u)})}{(ua^{-t}-b^t)^r}e^{rxt}=\sum_{n=0}^{\infty}H_n^{(k_1,\ldots,k_r)}(x;u,a,b)\frac{t^n}{n!},\quad |t|<\frac{2\pi}{|\ln a+\ln b|} \tag{45}$$

$$\frac{Li_{(k_1,\ldots,k_r)}(1-e^{(1-u)})}{(ua^{-t}-b^t)^r}c^{rxt}=\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r)}(x;u,a,b,c)\frac{t^n}{n!},\quad |t|<\frac{2\pi}{|\ln a+\ln b|} \tag{46}$$

§2. Main Theorems

In this section, we introduce our main results. We give some theorems and corollaries which are related to generalized Multi poly-Bernoulli numbers and generalized Multi poly-Euler polynomials. We present some recurrence formulae for generalized Multi-poly-Bernoulli and Euler polynomials.

Theorem 2.1 *Let $a,b>0$ and $a\neq b$, we have*

$$B_n^{(k_1,\ldots,k_r)}(a,b)=B_n^{(k_1,\ldots,k_r)}\left(\frac{-\ln b}{\ln a+\ln b}\right)(\ln a+\ln b)^n \tag{47}$$

proof By applying Definition 1.4, we have

$$\frac{Li_{(k_1,\ldots,k_r)}(1-(ab)^{-x})}{(b^x-a^{-x})^r}=\sum_{n=0}^{\infty}\frac{B_n^{(k_1,\ldots,k_r)}(a,b)}{n!}x^n$$

$$\begin{aligned}\frac{Li_{(k_1,\ldots,k_r)}(1-(ab)^{-x})}{(b^x-a^{-x})^r}&=\frac{1}{b^{xr}}\left(\frac{Li_{(k_1,\ldots,k_r)}(1-e^{-x\ln ab})}{(1-e^{-x\ln ab})^r}\right)\\&=e^{-xr\ln b}\left(\frac{Li_{(k_1,\ldots,k_r)}(1-e^{-x\ln ab})}{(1-e^{-x\ln ab})^r}\right)\end{aligned}$$

So, we get

$$\frac{Li_{(k_1,\ldots,k_r)}(1-e^{-x\ln ab})}{(b^x-a^{-x})^r}=\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r)}\left(\frac{-\ln b}{\ln a+\ln b}\right)(\ln a+\ln b)^n\frac{x^n}{n!}$$

Therefore, by comparing the coefficients of $\frac{t^n}{n!}$ on both sides, proof will be complete

$$B_n^{(k_1,\ldots,k_r)}(a,b)=B_n^{(k_1,\ldots,k_r)}\left(\frac{-\ln b}{\ln a+\ln b}\right)(\ln a+\ln b)^n$$

□

The generalized Multi poly-Bernoulli and Euler numbers process a number of interesting properties which we state here

Theorem 2.2 *Let $a,b>0$ and $a\neq b$. For real algebraic u we have*

$$H_n^{(k_1,\ldots,k_r)}(u;a,b)=H_n^{(k_1,\ldots,k_r)}\left(u;\frac{\ln a}{\ln a+\ln b}\right)(\ln a+\ln b)^n. \tag{48}$$

Next, we investigate a strong relationships between $B_n^{(k_1,\ldots,k_r)}(a,b)$ and $B_n^{(k_1,\ldots,k_r)}$.

Theorem 2.3 *Let $a,b>0$,$a\neq b$and $a>b>0$, we have*

$$B_n^{(k_1,\ldots,k_r)}(a,b)=\sum_{i=0}^{j}(-r)^{j-i}(\ln a+\ln b)^i(\ln b)^{j-i}\binom{j}{i}B_i^{(k_1,\ldots,k_r)}. \tag{49}$$

□

By applying Definition 1.4, we have

$$\begin{aligned}\frac{Li_{(k_1,\ldots,k_r)}(1-(ab)^{-x})}{(b^x-a^{-x})^r}&=\frac{1}{b^{xr}}\frac{Li_{(k_1,\ldots,k_r)}(1-(ab)^{-x})}{(1-e^{-x\ln ab})^r}\\&=\left(\sum_{k=0}^{\infty}\frac{(\ln b)^k}{k!}x^kr^k(-1)^k\right)\left(\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_r)}(\ln a+\ln b)^n\frac{x^n}{n!}\right)\\&=\sum_{j=0}^{\infty}\left(\sum_{i=0}^{j}(-r)^{j-i}\frac{B_j^{(k_1,\ldots,k_r)}(\ln a+\ln b)^i(\ln b)^{j-i}}{i!(j-i)!}x^j\right)\end{aligned}$$

By comparing the coefficient of $\frac{t^n}{n!}$ on both sides, we get.

$$B_j^{(k_1,\ldots,k_r)}(a,b)=\sum_{i=0}^{j}(-r)^{j-i}(\ln a+\ln b)^i(\ln b)^{j-i}\binom{j}{i}B_i^{(k_1,\ldots,k_r)}$$

□

By the same method proceeded in the proof of Theorem 2.3, we obtained similar relations for $H_n^{(k_1,\ldots,k_r)}(u;a,b)$ and $H_n^{(k_1,\ldots,k_r)}$.

Theorem 2.4 *Let $a, b > 0$, and $b > a > 0$. For algebraic real number u, we have*

$$H_n^{(k_1,\ldots,k_r)}(u; a, b) = \sum_{i=0}^{n} r^i (\ln a + \ln b)^i (\ln a)^{n-i} \binom{n}{i} H_i^{(k_1,\ldots,k_r)} \tag{50}$$

Theorem 2.5 *Let $x \in R$ and conditions of Theorem 2.3 holds true, then we get*

$$B_n^{(k_1,\ldots,k_r)}(x; a, b, c) = \sum_{l=0}^{n} \binom{n}{l} r^{n-l} (\ln c)^{n-l} B_l^{(k_1,\ldots,k_r)}(a, b) x^{n-l} \tag{51}$$

$$H_n^{(k_1,\ldots,k_r)}(u; x, a, b, c) = \sum_{l=0}^{n} \binom{n}{l} r^{n-l} (\ln c)^{n-l} H_l^{(k_1,\ldots,k_r)}(u; a, b) x^{n-l} \tag{52}$$

Proof By applying Definitions 1.4 and 1.5, proof will be complete. □

Theorem 2.6 *Let conditions of Theorem 2.5 holds true, we obtain*

$$H_n^{(k_1,\ldots,k_r)}(u; x, a, b, c) = \sum_{k=0}^{n} \binom{n}{k} r^{n-k} (\ln c)^{n-k} H_k^{(k_1,\ldots,k_r)}(u, \frac{\ln a}{\ln a + \ln b})(\ln a + \ln b)^k x^{n-k} \tag{53}$$

$$B_n^{(k_1,\ldots,k_r)}(x; a, b, c) = \sum_{k=0}^{n} \binom{n}{k} r^{n-k} (\ln c)^{n-k} B_k^{(k_1,\ldots,k_r)}(\frac{-\ln b}{\ln a + \ln b})(\ln a + \ln b)^k x^{n-k}. \tag{54}$$

Proof By applying Theorems 2.1 and 2.5, we get (53), and Obviously, the result of (54) is similar with (53). □

Theorem 2.7 *Let conditions of Theorem 2.5 holds true, then we get*

$$B_n^{(k_1,\ldots,k_r)}(x; a, b, c) = \tag{55}$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{k-j} \binom{n}{k} \binom{k}{j} r^{n-k} (\ln c)^{n-k} (\ln b)^{k-j} (\ln a + \ln b)^j B_j^{(k_1,\ldots,k_r)} x^{n-k}$$

$$H_n^{(k_1,\ldots,k_r)}(x; a, b, c) = \tag{56}$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} r^{n-k} (\ln c)^{n-k} (\ln a)^{k-j} (\ln c + \ln b)^j H_j^{(k_1,\ldots,k_r)} x^{n-k}$$

$$B_n^{(k_1,\ldots,k_r)}(x+1; a, b, c) = B_n^{(k_1,\ldots,k_r)}(x; ac, \frac{b}{c}, c) \tag{57}$$

$$H_n^{(k_1,\ldots,k_r)}(u, 1-x, ac, b, c) = B_n^{(k_1,\ldots,k_r)}(u, -x, ac, \frac{b}{c}, c) \tag{58}$$

$$B_n^{(k_1,\ldots,k_r)}(x+y; a, b, c) = \sum_{k=0}^{n} \binom{n}{k} r^{n-k} (\ln c)^{n-k} B_n^{(k_1,\ldots,k_r)}(x; a, b, c) y^{n-k} \tag{59}$$

$$H_n^{(k_1,\ldots,k_r)}(u; x+y, a, b, c) = \sum_{k=0}^{n} \binom{n}{k} r^{n-k} (\ln c)^{n-k} H_n^{(k_1,\ldots,k_r)}(x; a, b, c) y^{n-k} \tag{60}$$

Proof We only prove (59) and (55)-(60) can be derived by Definitions 1.4 and 1.5.

$$\frac{Li_{(k_1,\ldots,k_r)}(1-(ab)^{-t})}{(b^t-a^{-t})^r}c^{(x+y)rt} = \sum_{n=0}^{\infty} B_n^{(k_1,\ldots,k_r)}(x+y,a,b,c)\frac{t^n}{n!}$$
$$= \frac{Li_{(k_1,\ldots,k_r)}(1-(ab)^{-t})}{(b^t-a^{-t})^r}c^{xrt}.c^{yrt}$$
$$= \left(\sum_{n=0}^{\infty} B_n^{(k_1,\ldots,k_r)}(x;a,b,c)\frac{t^n}{n!}\right)\left(\sum_{i=0}^{n}\frac{y^i(\ln c)^i r^i}{i!}t^i\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}r^{n-k}y^{n-k}(\ln c)^{n-k}B_k^{(k_1,\ldots,k_r)}(x;a,b,c)\right)\frac{t^n}{n!}$$

So by comparing the coefficients of $\frac{t^n}{n!}$ in the two expressions, we obtain the desired result 2.13.
□

Theorem 2.8 *By the same method proceeded in the proof of previous Theorems, we find similar relations for $B_n^{(k_1,\ldots,k_r)}(t)$ and $H_n^{(k_1,\ldots,k_r)}(u,t)$.*

$$B_n^{(k_1,\ldots,k_r)}(t) = B_n^{(k_1,\ldots,k_r)}(e^{1+t},e^{-t}) \tag{61}$$
$$H_n^{(k_1,\ldots,k_r)}(u,t) = H_n^{(k_1,\ldots,k_r)}(u;e^{t},e^{1-t}) \tag{62}$$

Now, we present formulae which show a deeper motivation of generalized poly-Bernoulli and Euler polynomials.

Theorem 2.9 *Let $x,y\in R$ and conditions of Theorem 2.5 holds true, we get*

$$B_n^{(k_1,\ldots,k_r)}(x,a,b,c) = (\ln a+\ln b)^n B_n^{(k_1,\ldots,k_r)}\left(\frac{-\ln b+x\ln c}{\ln a+\ln b}\right) \tag{63}$$
$$H_n^{(k_1,\ldots,k_r)}(u;x,a,b,c) = H_n^{(k_1,\ldots,k_r)}\left(u;\frac{\ln a+x\ln c}{\ln a+\ln b}\right) \tag{64}$$

Proof We can write

$$\sum_{n=0}^{\infty} B_n^{(k_1,\ldots,k_r)}(x;a,b,c)\frac{t^n}{n!} = \frac{Li_{(k_1,\ldots,k_r)}(1-(ab)^{-t})}{(b^t-a^{-t})^r}c^{xrt}$$
$$= \frac{1}{b^{rt}}\frac{Li_{(k_1,\ldots,k_r)}(1-(ab)^{-t})}{(1-(ab)^{-t})^r}c^{xrt}$$
$$= e^{r(-\ln b+x\ln c)t}\left(\frac{Li_{(k_1,\ldots,k_r)}(1-e^{-t\ln ab})}{(1-e^{-t\ln ab})^r}\right)$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we get

$$B_n^{(k_1,\ldots,k_r)}(x;a,b,c) = (\ln a+\ln b)^n B_n^{(k_1,\ldots,k_r)}\left(\frac{-\ln b+x\ln c}{\ln a+\ln b}\right).$$

□

GI-Sang Cheon and H.M.Srivastava in [8],[10] investigated the classical relationship between Bernoulli and Euler polynomials . Now we present a relationship between generalized Multi poly-Bernoulli and generalized Euler polynomials. The following relation (65) are given by Q.M.Luo, So by applying this recurrence formula, we obtain Theorem 2.10,

$$E_k(x+1,1,b,b)+E_k(x,1,b,b)=2x^k(\ln b)^k \tag{65}$$

Theorem 2.10 *Let $a,b>0$, we have*

$$B_n^{(k_1,\ldots,k_r)}(x+y;a,b)= \tag{66}$$
$$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left[B_k^{(k_1,\ldots,k_r)}(y,a,b)+B_k^{(k_1,\ldots,k_r)}(y+1,a,b)\right]r^{n-k}E_{n-k}(x,1,b,b)$$

Proof We know

$$B_n^{(k_1,\ldots,k_r)}(x+y;1,b,b)=\sum_{k=0}^{n}\binom{n}{k}r^{n-k}(\ln b)^{n-k}B_k^{(k_1,\ldots,k_r)}(y;1,b,b)x^{n-k},$$
$$E_k(x+y,1,b,b)+E_k(x,1,,b,b)=2x^k(\ln b)^k$$

So, we obtain

$$B_n^{(k_1,\ldots,k_r)}(x+y,1,b,b)=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}r^{n-k}(\ln b)^{n-k}B_k^{(k_1,\ldots,k_r)}(y;1,b,b)\times$$
$$\left[\frac{1}{(\ln b)^{n-k}}(E_{n-k}(x;1,b,b)+E_{n-k}(x+1,1,b,b))\right]$$
$$=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}r^{n-k}B_k^{(k_1,\ldots,k_r)}(y;1,b,b)\times$$
$$\left[(E_{n-k}(x;1,b,b)+\sum_{j=0}^{n-k}\binom{n-k}{j}E_j(x,1,b,b))\right]$$

$$=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}r^{n-k}B_k^{(k_1,\ldots,k_r)}(y;1,b,b)E_{n-k}(x;1,b,b)$$
$$+\frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}r^{n-k}E_j(x;1,b,b)\sum_{k=0}^{n-j}\binom{n-j}{k}B_k^{(k_1,\ldots,k_r)}(y;1,b,b)$$
$$=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}r^{n-k}B_k^{(k_1,\ldots,k_r)}(y;1,b,b)E_{n-k}(x;1,b,b)$$
$$+\frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}r^{n-k}B_{n-j}^{(k_1,\ldots,k_r)}(y+1;1,b,b)E_j(x;1,b,b)$$

So we have

$$\begin{aligned}&B_n^{(k_1,\ldots,k_r)}(x+y;a,b)\\&=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left[B_k^{(k_1,\ldots,k_r)}(y,a,b)+B_k^{(k_1,\ldots,k_r)}(y+1,a,b)\right]r^{n-k}E_{n-k}(x,1,b,b)\end{aligned}$$

Therefore we obtain the desired result (66). □

The following corollary is a straightforward consequence of Theorem 2.10.

Corollary 2.1(see [8],[10]) *In Theorem 2.10, if we set $r=1$, $k=1$ and $b=e$, we obtain*

$$B_n(x)=\sum_{\substack{k=0\\k\neq 1}}^{n}\binom{n}{k}B_kE_{n-k}(x). \tag{67}$$

Further work: In [25], Jang et al. gave new formulae on Genocchi numbers. They defined poly-Genocchi numbers to give the relation between Genocchi numbers, Euler numbers, and poly-Genocchi numbers. After Y. Simsek [26], gave a new generating functions which produce Genocchi zeta functions. So by applying a similar method of Kim-Kim [4], we can introduce generalized Genocchi Zeta functions and next define Multi poly-Genocchi numbers and obtain several properties in this area.

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Bounds on the Largest of Minimum Degree Laplician Eigenvalues of a Graph

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Abstract: In this paper we give three upper bounds for the largest of minimum degree Laplacian eigenvalues of a graph and also obtain a lower bound for the same.

Key Words: Minimum degree matrix, minimum degree Laplacian eigenvalues.

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§1. Introduction

Let $G = (V, E)$ be a simple, connected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. Assume that the vertices are ordered such that $d_1 \geq d_2 \geq \ldots \geq d_n$, where d_i is the degree of v_i for $i = 1, 2, \ldots, n$. The energy of G was first defined by I.Gutman [5] in 1978 as the sum of the absolute values of its eigenvalues. The energy of a graph has close links to Chemistry(see for instance [6]). The $n \times n$ matrix $m(G) = (d_{ij})$ is called the minimum degree matrix of G, where

$$d_{ij} = \begin{cases} min\{d_i, d_j\} & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

This was introduced and studied in [1]. The characteristic polynomial of the minimum degree matrix $m(G)$ is defined by

$$\begin{aligned} \phi(G; \lambda) &= det(\lambda I - m(G)) \\ &= \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n, \end{aligned} \tag{1.1}$$

where I is the unit matrix of order n. The minimum degree Laplacian matrix of G is $L(G) = D(G) - m(G)$, where $D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$. $L(G)$ is a real, symmetric matrix. The minimum degree Laplacian eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of the graph G, assumed in the non increasing order, are the eigenvalues of $L(G)$. The Laplacian matrix of G is $L_1(G) = D(G) - A(G)$, where $A(G)$ is the adjacency matrix of G. The eigenvalues of the laplacian matrix $L_1(G)$ are important in graph theory, because they have relations to numerous graph invariants including connectivity, expanding property, isoperimetric number, independence number, genus, diameter, mean distance, and bandwidth-type parameters of a graph(see, for example, [2,3,9,10]). In

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many applications one needs good bounds for the largest Laplacian eigenvalue (see for instance [2,3,9,10]). In this paper, we give three upper bounds and a lower bound for μ_1 the largest of minimum degree Laplacian eigenvalues of a graph.

§2. Main Results

In this section, we will give three upper bounds for μ_1 the largest of minimum degree Laplacian eigenvalues of a graph. We employ the following theorem to prove one of our main results.

Theorem 2.1([4]) *Let G be a simple graph with n vertices and m edges, and let $\Pi = (d_1, d_2, \ldots, d_n)$ be the degree sequence of G. Then,*

$$d_1^2 + d_2^2 + \ldots + d_n^2 \leq m(\frac{2m}{n-1} + n - 2).$$

Theorem 2.2 *Let G be a connected graph with n vertices and m edges. Then*

$$\mu_1 \leq \frac{2m + \sqrt{(n-1)\left[n(2|c_2| + m(\frac{2m}{n-1} + n - 2)) - 4m^2\right]}}{n},$$

where c_2 is the coefficient of λ^{n-2} in $det(\lambda I - m(G))$.

Proof Clearly

$$\mu_1 + \mu_2 + \ldots + \mu_n = Trace[L(G)] = \sum_{v \in V(G)} d_v, \tag{2.1}$$

$$\mu_1^2 + \mu_2^2 + ... + \mu_n^2 = 2|c_2| + \sum_{i=1}^{n} d_i^2. \tag{2.2}$$

By Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right). \tag{2.3}$$

Putting $a_i = 1$ and $b_i = \mu_i$ for $i = 2, ..., n$ in (2.3), we get

$$\left(\sum_{i=1}^{n} \mu_i - \mu_1\right)^2 \leq (n-1)\left(\sum_{i=1}^{n} \mu_i^2 - \mu_1^2\right).$$

Using (2.1) and (2.2) in above inequality, we obtain

$$\left(\sum_{v \in V(G)} d_v - \mu_1\right)^2 \leq (n-1)\left[2|c_2| + \sum_{i=1}^{n} d_i^2\right] - (n-1)\mu_1^2.$$

After some simplifications, we deduce that

$$\left(n\mu_1 - \sum_{v \in V(G)} d_v\right)^2 + (n-1)\left(\sum_{v \in V(G)} d_v\right)^2 \leq n(n-1)\left[2|c_2| + \sum_{i=1}^{n} d_i^2\right].$$

$$\text{i.e.,} \qquad n\mu_1 - \sum_{v\in V(G)} d_v \le \sqrt{(n-1)\left[n(2|c_2| + \sum_{i=1}^{n} d_i^2) - \left(\sum_{i=1}^{n} d_i\right)^2\right]}.$$

Therefore

$$\mu_1 \le \frac{\sum_{i=1}^{n} d_i + \sqrt{(n-1)\left[n\left(2|c_2| + \sum_{i=1}^{n} d_i^2\right) - \left(\sum_{i=1}^{n} d_i\right)^2\right]}}{n}. \tag{2.4}$$

Employing Theorem 2.1 and $\sum_{i=1}^{n} d_i = 2m$ in (2.4), we see that

$$\mu_1 \le \frac{2m + \sqrt{(n-1)\left[n(2|c_2| + m(\frac{2m}{n-1} + n - 2)) - 4m^2\right]}}{n}.$$

This completes the proof. □

The following theorem gives another type of upper bound for μ_1.

Theorem 2.3 *Let G be connected graph with n vertices and m edges. Then*

$$\mu_1 \le \sqrt{2d_1^2 + 4m - 2d_n^3(n - d_1)}.$$

Proof Suppose that $X = (x_1, x_2, x_3, \ldots, x_n)^T$ be an eigenvector with unit length corresponding to μ_1. Then

$$L(G)X = \mu_1 X.$$

Hence, for $u \in V(G)$,

$$\mu_1 x_u = d_u x_u - \sum_{\substack{v \in V(G) \\ v \ne u}} d_{uv} x_v.$$

Here x_u we mean x_i if $u = v_i$. Therefore

$$\mu_1 x_u = \sum_{vu\in E(G)} (x_u - min(d_u, d_v)x_v). \tag{2.5}$$

By Cauchy-Schwarz inequality, we have

$$\mu_1^2 x_u^2 \le \left(\sum_{vu\in E(G)} 1^2\right)\left(\sum_{vu\in E(G)} (x_u - min(d_u, d_v)x_v)^2\right)$$

$$= d_u\left[\sum_{vu\in E(G)} x_u^2 + \sum_{vu\in E(G)} min(d_u, d_v)^2 x_v^2 - 2x_u min(d_u, d_v)x_v\right].$$

Observe that

$$-2x_u \sum_{vu\in E(G)} min(d_u, d_v)x_v \le d_u x_u^2 + \sum_{vu\in E(G)} min(d_u, d_v)^2 x_v^2. \tag{2.6}$$

Hence,

$$\mu_1^2 x_u^2 \leq d_u \left[\sum_{vu\in E(G)} x_u^2 + \sum_{vu\in E(G)} min(d_u, d_v)^2 x_v^2 + d_u x_u^2 + \sum_{vu\in E(G)} min(d_u, d_v)^2 x_v^2 \right].$$

i.e.,
$$\mu_1^2 x_u^2 \leq 2d_u^2 x_u^2 + 2d_u \sum_{vu\in E(G)} min(d_u, d_v)^2 x_v^2. \tag{2.7}$$

Consequently,

$$\begin{aligned}\mu_1^2 &= \mu_1^2 \sum_{u\in V(G)} x_u^2 \\ &\leq \sum_{u\in V(G)} [2d_u^2 x_u^2 + 2d_u \sum_{vu\in E(G)} min(d_u, d_v)^2 x_v^2] \\ &= 2 \sum_{u\in V(G)} d_u^2 x_u^2 + 2 \sum_{u\in V(G)} d_u \sum_{vu\in E(G)} min(d_u, d_v)^2 x_v^2.\end{aligned}$$

Thus

$$\mu_1^2 \leq 2d_1^2 + 2 \sum_{u\in V(G)} d_u \sum_{vu\in E(G)} min(d_u, d_v)^2 x_v^2. \tag{2.8}$$

Now let $v \nsim u$ mean that u and v are not adjacent. Then

$$\begin{aligned}&\sum_{u\in V(G)} d_u \sum_{vu\in E(G)} min(d_u, d_v)^2 x_v^2 \\ &= \sum_{u\in V(G)} d_u [1 - \sum_{v\nsim u} min(d_u, d_v)^2 x_v^2] = 2m - \sum_{u\in V(G)} d_u \sum_{v\nsim u} min(d_u, d_v)^2 x_v^2 \\ &= 2m - \left(\sum_{u\in V(G)} d_u min(d_u, d_v)^2 x_u^2 + \sum_{u\in V(G)} d_u \sum_{v\nsim u, v\neq u} min(d_u, d_v)^2 x_v^2 \right) \\ &\leq 2m - \left(d_n^2 \sum_{u\in V(G)} d_u x_u^2 + \sum_{u\in V(G)} d_n \sum_{v\nsim u, v\neq u} d_n^2 x_v^2 \right) \\ &= 2m - \left(d_n^2 \sum_{u\in V(G)} d_u x_u^2 + \sum_{u\in V(G)} d_n^3 (n - d_u - 1) x_u^2 \right) \\ &= 2m - \left(d_n^2 \sum_{u\in V(G)} d_u x_u^2 + d_n^3 \sum_{u\in V(G)} n x_u^2 - d_n^3 \sum_{u\in V(G)} d_u x_u^2 - d_n^3 \sum_{u\in V(G)} x_u^2 \right) \\ &\leq 2m - d_n^3 \sum_{u\in V(G)} (n - d_1) x_u^2 \\ &= 2m - d_n^3 (n - d_1).\end{aligned}$$

Hence, employing this in (2.8) we have

$$\mu_1^2 \leq 2d_1^2 + 4m - 2d_n^3 (n - d_1).$$

Therefore

$$\mu_1 \le \sqrt{2d_1^2 + 4m - 2d_n^3(n - d_1)}.$$

Theorem 2.4 *Let G be a connected graph then*

$$\mu_1 \le max\left(\sqrt{2\left(d_u^2 + d_1^2 m_u d_u\right)} \;\;:\;\; u \in V(G)\right).$$

Proof From (2.7) we have

$$\mu_1^2 x_u^2 \le 2d_u^2 x_u^2 + 2d_u \sum_{vu \in E(G)} min(d_u, d_v)^2 x_v^2.$$

Thus

$$\begin{aligned}
\mu_1^2 \sum_{u\in V(G)} x_u^2 &\le 2 \sum_{u\in V(G)} d_u^2 x_u^2 + 2 \sum_{u\in V(G)} d_u \sum_{vu\in E(G)} min(d_u, d_v)^2 x_v^2. \\
&\le 2 \sum_{u\in V(G)} d_u^2 x_u^2 + 2d_1^2 \sum_{u\in V(G)} d_u \sum_{vu\in E(G)} x_v^2 \\
&= 2\left[\sum_{u\in V(G)} d_u^2 x_u^2 + d_1^2 \sum_{u\in V(G)} x_u^2 \sum_{vu\in E(G)} d_v\right] \\
&= 2\left[\sum_{u\in V(G)} d_u^2 x_u^2 + d_1^2 \sum_{u\in V(G)} x_u^2 m_u d_u\right]
\end{aligned}$$

where m_u = average degree of the vertices adjacent to u.
So,

$$\mu_1 \le \sqrt{2 \sum_{u\in V(G)} \left(d_u^2 + d_1^2 m_u d_u\right) x_u^2}.$$

Hence

$$\mu_1 \le \max\left\{\sqrt{2\left(d_u^2 + d_1^2 m_u d_u\right)} \;\;:\;\; u \in V(G)\right\}.$$

§3. Lower Bonud for Spectral Radius of Graphs

In this section we establish a lower bound for the spectral radius μ_1 of G.

Lemma 3.1([7][8]) *Let M be real symmetric matrix with eigenvalues $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$. Given a partition $\{1, 2, \ldots, n\} = \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_m$ with $|\Delta_i| = n_i > 0$, consider the corresponding blocking $M = (M_{ij})$, so that M_{ij} is an $n_i \times n_j$ block. Let e_{ij} be the sum of the entries in M_{ij} and put $B = (\frac{e_{ij}}{n_i})$ i.e.,($\frac{e_{ij}}{n_i}$ is an average row sum in M_{ij}). let $\gamma_1 \ge \gamma_2 \ge \ldots \ge \gamma_m$ be the eigenvalues of B. Then the inequalities*

$$\lambda_i \ge \gamma_i \ge \lambda_{n-m+i} \qquad (i = 1, 2, \ldots, m)$$

hold. Moreover, if for some integer k, $1 \le k \le m, \lambda_i = \gamma_i$ for $i = 1, 2, \ldots, k$ and $\lambda_{n-m+i} = \gamma_i$ for $i = k+1, k+2, \ldots, m$, then all the blocks M_{ij} have constant row and column sums.

Let G be a connected graph with n vertices and m edges. Let $V_1 = \{v_1, v_2, \ldots v_{n_1}\}$ and $V_2 = \{v_{n_1+1}, v_{n_1+2} \ldots v_n\}$ be two partitions of vertices of graph G. Let

$$r_1 = \frac{1}{n_1} \sum_{\substack{i,\,j\,=\,1 \\ i\,\neq\,j}}^{n_1} min(d(v_i), d(v_j)), \quad r_2 = \frac{1}{n-n_1} \sum_{\substack{i,\,j\,=\,1 \\ i\,\neq\,j}}^{n-n_1} min(d(v_{n_1+i}), d(v_{n_1+j})),$$

$$k_1 = \frac{-1}{n_1} \sum_{\substack{i,\,j\,=\,1 \\ i\,\neq\,j}}^{n-n_1} min(d(v_i), d(v_{n_1+j})), \quad k_2 = \frac{-1}{n-n_1} \sum_{\substack{i\,=\,1 \\ j\,=\,1,2,\ldots,n \\ i\,\neq\,j}}^{n-n_1} min(d(v_{n_1+i}), d(v_j)),$$

$$d_1 = \frac{1}{n_1} \sum_{v \in V_1} d(v), \qquad d_2 = \frac{1}{n-n_1} \sum_{v \in V_2} d(v),$$

where $d(v)$ is the degree of the vertex v of G. Now we prove the following theorem.

Theorem 3.2 *Let G be a connected graph with n vertices and m edges, then*

$$\mu_1 \geq \frac{1}{2}\{d_2 + d_1 - r_2 - r_1 + \sqrt{(d_2 - d_1 - r_2 + r_1)^2 - 4k_1k_2}\}.$$

Proof Rewrite $L(G)$ as

$$L(G) = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}.$$

For $1 \leq i,\, j \leq 2$, let e_{ij} be the sum of the entries in L_{ij} and put $B = (e_{ij}/n_i)$. Then

$$B = \begin{pmatrix} d_1 - r_1 & k_1 \\ k_2 & d_2 - r_2 \end{pmatrix},$$

and so

$$|\lambda I - B| = \begin{vmatrix} \lambda - (d_1 - r_1) & -k_1 \\ -k_2 & \lambda - (d_2 - r_2) \end{vmatrix}.$$

Therefore we have

$$\lambda = \frac{1}{2}\{d_2 + d_1 - r_2 - r_1 \pm \sqrt{(d_2 - d_1 - r_2 + r_1)^2 - 4k_1k_2}\}.$$

Thus by Lemma 3.1 we get

$$\mu_1 \geq \frac{1}{2}\{d_2 + d_1 - r_2 - r_1 + \sqrt{(d_2 - d_1 - r_2 + r_1)^2 - 4k_1k_2}\}. \qquad \square$$

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I want to bring out the secrets of nature and apply them for the happiness of man. I don't know of any better service to offer for the short time we are in the world.

By Thomas Edison, an American inventor.

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